Ramsey number of nK_4

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1 Introduction

Ramsey's Theorem is a foundational result in combinatorial mathematics which guarantees the presence of a special substructures in any structure of a large enough size. This theorem is usually mentioned in the context of graph theory, where it states that for any two positive integers s and t, there exists a number N such that for $n \geq N$ any red-blue coloring of the edges of the complete graph on n vertices has a red clique containing s vertices or a blue clique containing t vertices. For fixed s and t, the Ramsey number $R(s,t)$ is the minimum value of N for which the statement holds. Ramsey numbers are known for certain classes of graphs, but they quickly become very difficult, if not impossible, to calculate exactly since the number of colorings on a graph of size n is $2^{\binom{n}{2}}$, which grows extremely fast.

One class of graphs of which there has been success calculating Ramsey numbers is the Ramsey number of multiple copies of a graph. Ramsey's theorem can be generalized from cliques to general graphs by defining the Ramsey number of two graphs G and H, written $R(G,H)$ to be the minimal integer N such that any red-blue edge coloring of a complete graph on N vertices must contain a copy of G using only red edges or H using only blue edges. Similarly, we define $R(nG,nH)$ to be the minimal integer N such that any red-blue edge coloring of a complete graph on N vertices must contain n vertex-disjoint copies of G using only red edges or n vertex-disjoint copies of H using only blue edges. Burr, Erdös, and Spencer [2] proved in 1975 that for n_0 sufficiently large, there exists a constant c such that for connected graphs G,H and $n \geq n_0$ we have $R(nG,nG) = (2k-i)n + c$ where k is the number of vertices in G and i is the largest size of an independent set in G. While this theorem provides insight on the long term behavior of multiple-copy Ramsey numbers, the upper bound on n_0 is very high, originally triple exponential in k from [2] but recently lowered to single exponential in k from [1]. The value of n_0 has been found for certain classes of graphs, for example the size of a graph needed to find n vertex disjoint triangles of the same color satisfies $R(nK_3) = 5n$ for $n \geq 2$ [2]. A natural follow-up question is to find the Ramsey number of multiple copies K_4, the clique on four vertices. The result $R(K_4) = 18$ is well established, but little progress has been made discovering $R(nK_4)$ for small values or finding the value of n_0 where $R(nK_4)$ converges to its long-term behavior. This independent work aimed to provide new discoveries and insight on both of these areas. While this independent work was unsuccessful in answering the ultimate of question of finding $R(nK_4,nK_4)$ for all n, it was successful in finding many intermediate results which make significant progress to answering this question, and in this paper we provide the proof of the first intermediate result, $R(nK_4,nK_4) = 20$ by showing both a lower and an upper bound of 20. To prove a lower bound of 20, we will show that there exists a coloring of a complete graph on 20 vertices such that it does not have a red $2K_4$ or a blue K_4, and to show an upper bound of 20, we will argue that any red-blue coloring of a complete graph on 20 vertices must have a red $2K_4$ or a blue K_4.

2 Lower Bound: $R(2K_4, K_4) = 20$

Observation 1. $R(2K_4, K_4) \geq 20$

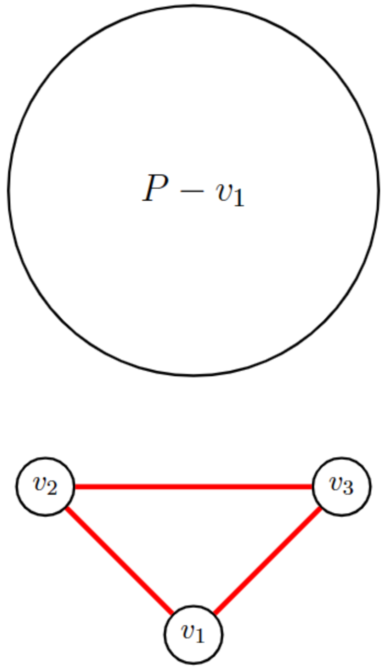

Figure 1: Description of the lower bound graph showing $R(2K_4, K_4) \geq 20$. This graph has 19 vertices without any red $2K_4$ and without any blue K_4. $P - v_1$ together with either v_1, v_2 or v_3 forms the Paley graph, which is the single 17 vertex graph without a monochromatic K_4.

Proof. Since $R(K_4, K_4) = 18$, we can create a graph P on 17 vertices that does not contain any K_4 (the unique Paley graph). Let v_1 be an arbitrary vertex of P and form the graph G by adding two additional vertices v_2, v_3 to P such that v_2 and v_3 are connected to P in the same manner as v_1 and the edges between v_1, v_2, and v_3 are all red. Since P does not contain a monochromatic K_4, we know that any monochromatic K_4 must use two of v_1, v_2, v_3. Thus G must not have a red $2K_4$ because any two K_4 must intersect in v_1, v_2, or v_3, and G must not have a blue K_4 because the edges between v_1, v_2, and v_3 are all red. Therefore G is a 19 vertex graph with no red $2K_4$ and no blue K_4. $\square$

3 Upper Bound

In this section, we assume (for sake of contradiction) that there exists a graph G which is a complete graph on 20 vertices whose edges are colored blue or red such that G does not contain a red $2K_4$ or a blue K_4. Additionally, let $K_5 - e$ represent the graph formed by removing one edge from a K_5, and in particular, let H represent the specific coloring of a K_5 which contains a red $K_5 - e$ and one blue edge.

Theorem 2. *G contains a copy of H.*

Proof. Since $R(K_5 - e, K_4) = 19$, we know that among any 19 vertices of G, we can find a red $K_5 - e$. Assume that G does not contain a copy of H, so in all such $K_5 - e$ the missing edge is red, creating a K_5. Let t be the size of the largest red clique in G. Since a red K_8 creates a red $2K_4$, we must have $5 \leq t \leq 7$, If any vertex sends 3 red edges to our red K_t, then this must create a red K_{t+1} if all other edges from this vertex are red or create a copy of H if the vertex sends at least one blue edge to K_t, which contradict the fact that the largest clique in G has size t and that G is H free, respectively. Thus each vertex can send at most two red edges to the K_t, so there are at least $(20-t)(t-2)$ blue edges connecting K_t to the rest of G. The average blue degree among vertices in K_t then is $(t-2)(20-t)/t$ which for all $5 \leq t \leq 7$ is greater than 8. Thus one vertex in K_t sends 9 blue edges to $G \setminus K_t$, and since $R(4,3) = 9$ this must lead to a blue K_4 or two red K_4, one in K_t and one in $G \setminus K_t$. Therefore we have proven that if G does not have a copy of H, it must have a red $2K_4$ or a blue K_4, which is a contradiction to the definition of G. □

Theorem 3. $R(2K_4, K_4) = 20$

Proof. Assume otherwise, that the counterexample graph G exists. From Theorem 1, we know that G must contain a copy of H. Let w_1, w_2 be the two vertices connected by the blue edge in H, and let v_1, v_2, v_3 be the other vertices of H. Additionally, partition our vertices into two sets: $T = \{v_1, v_2, v_3\}$ which form a red triangle and P encompassing the remaining 17 vertices, and we claim that P must not contain any monochromatic K_4. To see this, P cannot contain a red K_4 because this would have to contain at most one of w_1, w_2 since they have a blue edge between them, allowing the unused vertex of w_1, w_2 together with v_1, v_2, v_3 to form a second K_4, and we know that P does not have a blue K_4 since G does not contain a blue K_4 by assumption. Since $R(4,3) = 9$, we can find a K_3 in either color among any 9 vertices of P. This implies that every vertex in P must have degree 8 in both colors towards other vertices in P and that all vertices in T red degree at least 9 towards P. Let R_1, R_2, R_3 be the red neighbors of v_1, v_2, v_3 respectively in P, and we know that $w_1, w_2 \in R_1 \cap R_2 \cap R_3$. We seek to further understand how R_1, R_2, R_3 intersect.

Claim 1. *Any red edge between vertices in $R_1 \cap R_2$ must intersect all red triangles in R_3 (in a vertex).*

Proof. If this were not the case, then the red triangle in R_3 together with v_3 form a red K_4, and the two vertices in $R_1 \cap R_2$ together with $v_1.v_2$ form a second vertex disjoint K_4. This contradicts the claim that G does not have a red $2K_4$ or a blue K_4. □

Claim 2. *If some $w \in R_1 \cap R_2 \cap R_3$ belongs to all red triangles in R_3, then w and v_3 must be joined to every vertex of P by edges of the same color.*

Proof. Let P' be formed from P by swapping w with v_3. We claim that there must not be any monochromatic K_4 among vertices in P'. This implies that each vertex has degree 8 in both colors in both P and P', so w and v_3 are connected to P with edges of the same colors. To see that there is not any monochromatic K_4 in P', we first know that G must not have any blue K_4 by assumption. If there were a red K_4 in P', it must contain v_3 and some $x_1, x_2, x_3 \in P$ form a red K_4, and thus x_1, x_3, x_3 form a red triangle in $R_3 - w$, which is a contradiction. □

We call this property mentioned in Claim 2 “connected to P with the same colors" for the remainder of the proof.

Corollary 4. *If there is a red edge between $x \in R_1 \cap R_2 \setminus R_3$ and $w \in R_1 \cap R_2 \cap R_3$, then w and v_3 must be joined to very vertex of P by edges of the same color.*

Proof. By Claim 1, w must belong to every red triangle of R_3, so Claim 2 applies. □

Claim 3. $|R_1 \cap R_2 \setminus R_3| \leq 1$

Proof. Let $x_1, x_2 \in R_1 \cap R_2 \setminus R_3$. Claim 1 tells us that x_1 and x_2 cannot be connected by a red edge because x_1 and x_2 are not in R_3. Thus x_1x_2 is blue, and one of the edges between x_1, x_2 and w_1, w_2 must be red or else we have a blue K_4. By Corollary 4, one of $w_1.w_2$ must be connected to P in the same manner as v_3, but this is impossible because v_3w_1, v_3w_2 are both red and w_1w_2 is blue. □

Since all of these claims are symmetric to R_1, R_2, R_3, we can use the inclusion-exclusion principle to show

$$|R_1 \cap R_2 \setminus R_3| + |R_1 \cap R_3 \setminus R_2| + |R_2 \cap R_3 \setminus R_1| + 2|R_1 \cap R_2 \cap R_3| = |R_1| + |R_2| + |R_3| - |R_1 \cup R_2 \cup R_3| \geq 3 * 9 - 17 = 10$$

Which implies that $|R_1 \cap R_2 \cap R_3| \geq 4$. We now seek to tighten the bounds on $|R_1 \cap R_2 \cap R_3|$ and $|R_1 \cap R_2 \setminus R_3|$.

Claim 4. $R_1 \cap R_2 \setminus R_3 = \emptyset$

Proof. Assume that there is an $x \in R_1 \cap R_2 \setminus R_3$. If x sends a red edge towards $w \in R_1 \cap R_2 \cap R_3$ then w must be connected to P in the same manner as v_3, but xw is red while xv_3 is blue since $x \notin R_3$. Thus all edges from x to $R_1 \cap R_2 \cap R_3$ must be blue, which means that $R_1 \cap R_2 \cap R_3$ must not have a blue triangle to ensure that there is no blue K_4.

Let w_3, w_4 be two other vertices in $R_1 \cap R_2 \cap R_3$ besides w_1 and w_2 which we already know are connected by a blue edge. To ensure that there is no blue triangle in $R_1 \cap R_2 \cap R_3$, there must be at least two red edges between w_1, w_2 and w_3, w_4. We claim that w_3, w_4 can have red degree at most 1 towards w_1, w_2. If two red edges come from w_3 (assume w_3w_1 and w_3w_2 are both red), then by Claim 1 every triangle of R_2 needs to intersect vertices in both of these edges. Since w_1w_2 is blue, a red triangle cannot contain both of these vertices, so w_3 must be contained in every red triangle of R_2. By Claim 2, w_3 and v_2 are connected to P with the same colors, but w_3x is blue while v_2x is red, which is a contradiction.

We claim that both w_1 and w_2 contain red edges to the remainder of $R_1 \cap R_2 \cap R_3$. If w_2 is connected in blue to the remainder of $R_1 \cap R_2 \cap R_3$, then we know that all remaining edges in $R_1 \cap R_2 \cap R_3$ must be red to prevent a blue triangle. By Claim 1, we know that any red triangle inside any R_i must contain two of w_1, w_3, w_4, so w_2 cannot be in a red triangle contained in any R_i. Using Claim 1 again, w_2 cannot have any red edges to a vertex in the intersection of two R_i. Additionally, w_2 must send at most two red edges to sets of the form $R_1 \setminus (R_2 \cup R_3)$. If w_2 sends three red edges to $R_1 \setminus (R_2 \cup R_3)$, then these red neighbors of w_2 must all be connected with blue edges to prevent a red triangle from forming since such a red triangle would not contain any of w_1, w_3, w_4, but this forms a blue triangle which combined with v_2 forms a blue K_4 since v_2 must be connected in blue to these vertices since they are not contained in R_2. Thus we have shown that w_2 can have red degree at most 6 towards $R_1 \cup R_2 \cup R_3$. Since each vertex in P has degree 8 in both colors, we must have $|P \setminus (R_1 \cup R_2 \cup R_3)| \geq 2$, so by the same inclusion-exclusion argument as above we get $|R_1 \cap R_2 \cap R_3| \geq 5$. Let w_5 be the fifth vertex in $R_1 \cap R_2 \cap R_3$. w_2w_5 must be blue and to prevent a red triangle, all other edges incident to w_5 must be red. Finally we must have that w_1, w_3, w_4, w_5 and w_2, v_1, v_2, v_3 make a $2K_4$, our desired contradiction.

The only remaining case must have two red edges between w_1, w_2 and the remainder of $R_1 \cap R_2 \cap R_3$ which do not intersect in a vertex. Without loss of generality assume these edges are w_1w_3 and w_2w_4, and we must necessarily have that w_1w_4 and w_2w_3 are both blue. By Claim 1, any red triangle in R_2 must intersect w_1w_3 or w_2w_4 in a vertex, which implies that w_3w_4 is necessarily red and that this edge is contained in every red triangle in R_2. Thus w_3 and v_2 must be connected to P with the same colors, but w_3x is blue and v_2x is red. $\square$

Thus the inclusion-exclusion argument from above implies that $|R_1 \cap R_2 \cap R_3| \geq 5$.

Claim 5. *There can be no vertex $w \in R_1 \cap R_2 \cap R_3$ which belongs to all red triangles of R_2.*

Proof. Assume otherwise that there exists $w \in R_1 \cap R_2 \cap R_3$ which belongs to every red triangle of R_2. In the at least four remaining vertices of $R_1 \cap R_2 \cap R_3$, there must be a red edge $w'w''$ to prevent a blue K_4, so w', w'', v_1, v_3 form a red K_4. By Claim 2, w and v_2 are connected to P with the same colors, so w must be conneceted in red to all of R_2. Since $|R_2| \geq 9$, w still has at least 6 neighbors in $R_2 \setminus \{w', w''\}$, so $R_2 \setminus \{w', w''\}$ must contain a red edge $x'x''$. Now v_2, w, x', x'' form a second vertex disjoint red K_4. $\square$

Claim 6. *Any red triangle in R_2 needs at least 2 vertices from $R_1 \cap R_2 \cap R_3$.*

Proof. Assume otherwise that there exists a red triangle in R_2 with only one vertex in $R_1 \cap R_2 \cap R_3$. This triangle together with v_2 forms a red K_4, and since there must be a red edge in the remaining four vertices of $R_1 \cap R_2 \cap R_3$, the vertices incident to this edge together with v_1, v_3 form a second vertex disjoint K_4. $\square$

Claim 7. *Any red triangle in R_2 must be completely within $R_1 \cap R_2 \cap R_3$*

Proof. Assume otherwise that there exists a red triangle in R_2 with exactly two vertices w_4, w_5 contained in $R_1 \cap R_2 \cap R_3$. There cannot be any red edges in the remainder of $R_1 \cap R_2 \cap R_3$ because these vertices would have to intersect the triangle by Claim 1, which is impossible. Thus we know that $|R_1 \cap R_2 \cap R_3| = 5$ to ensure no blue K_4, which then implies that the sets of the form $|R_1 \setminus (R_2 \cup R_3)$ must all be size 4. Now w_4 cannot send two red edges to w_1, w_2, w_3 because that would force it to be in every red triangle of R_2, which contradicts Claim 5. However, w_4 cannot send all blue edges to w_1, w_2, w_3 because that would form a blue K_4, so w_4 (and for the same reasons w_5 must send exactly one red edge to w_1, w_2, w_3). If w_4 and w_5 send red edges to different vertices, then both vertices must be contained in every red triangle of R_2, which is once again impossible by Claim 5. If w_4 and w_5 send red edges to the same vertex w_1, then w_2, w_3 must only send blue edges to the remainder of $R_1 \cap R_2 \cap R_3$. w_2 must have red degree 8 towards the rest of P, none of which goes to $R_1 \cap R_2 \cap R_3$, so at least one of the sets of the form $R_1 \setminus (R_2 \cup R_3)$ must receive at least three red edges. Assume without loss of generality that w_2 sends three red edges to $R_1 \setminus (R_2 \cup R_3)$. By Claim 6, we know that there cannot be a red triangle containing w_2 and two vertices in $R_1 \setminus (R_1 \cup R_3)$, so the red neighbors of w_2 in this set must form a blue triangle. This blue triangle together with v_3 forms a blue K_4. □

To conclude the overall proof, we know that there must be a red triangle in R_2 because $R(4,3) = 9$. By Claim 7, this red triangle must be contained in $R_1 \cap R_2 \cap R_3$. Claim 5 tells us that no vertex can be in all red triangles of R_2, so there must be at least one other red triangle in $R_1 \cap R_2 \cap R_3$. Claim 1 tells us that these two red triangles must intersect in an edge, and then Claim 5 tells us that there must be a third triangle which does not use a vertex of this shared edge. This must create a red K_4 in $R_1 \cap R_2 \cap R_3$, which contradicts our assumption that P does not have any monochromatic K_4, completing the proof. □

4 Later Results

After achieving this result, we were able to find several other Ramsey numbers for multiple copies of K_4 and find complete sets of lower-bound graphs computationally, which are shown in the following list. Regarding the lower bound graphs, the notation $(k;\ G;\ H)$ represents the set of all possible graphs on k vertices without a red G or a blue H. So far, we have been able to find $R(nK_4, nK_4)$ for up to $n = 3$, and we are optimistic that the Ramsey number for $R(nK_4, nK_4)$ will converge to its long term known behavior at $n = 4$, which suggests that the problem is close to being solved. These results will hopefully be presented in a later paper.

- $R(2K_4, K_4) = 20$
  - $(16;\ 2K_4;\ K_4)$ – 50,033,249 graphs
  - $(17;\ 2K_4;\ K_4)$ – 28,206 graphs
  - $(18;\ 2K_4;\ K_4)$ – 79 graphs
  - $(19;\ 2K_4;\ K_4)$ – 1 graph
- $R(3K_4, K_4) = 22$
- $R(nK_4, K_4) = 4n + 8$
- $R(2K_4, 2K_4) = 22$
  - $(21;\ 2K_4;\ 2K_4)$ – 2 graphs
- $R(3K_4, 2K_4) = 24$
- $R(3K_4, 3K_4) = 27$
- $R(\{K_5 - e, 2K_4\}, K_4) = 19$

References

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[2] Stefan A Burr, P Erdős, and Joel H Spencer, *Ramsey theorems for multiple copies of graphs*, Transactions of the American Mathematical Society **209** (1975), 87–99.