Towards a Unified Model of Quantum Computation

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Abstract

In this work we provide a unifying framework from which to discuss the measures of computational complexity associated with different models of quantum computation, based on the resources being considered when discussing equivalence between the models. Due to the disparity between and the lack of a comprehensive theory comparing these various models in the literature, we first present a reformulation of selected existing results under this framework. We then utilize insights gained from this framework, particularly from the case of bang-bang time-optimal control of the Schrödinger equation, to propose a new measure of complexity for a generic parameterized quantum circuit (PQC). We conclude with a discussion of the potential implications of this new measure, as well as an outline for future work to further explore the connections presented here.

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1 Introduction

Quantum computing has been proposed as a new paradigm for achieving computational advantages over classical computers [1]. In order to adequately discuss these proposals and potential advantages, however, it is important to have a precise definition for a quantum "advantage." To do so, one needs to be able to assess the complexity of a particular quantum procedure, usually attained by measuring the *resources* utilized by the procedure.

Unfortunately, theories of quantum complexity across different models are at the moment far less established than those for classical complexity, and a unified model to discuss the many models of quantum computation does not yet exist. In particular, the fundamental barrier between discrete and continuous models has yet to be successfully broken (perhaps due to the innate property of *discretization* in quantum mechanics, though this is a topic better suited for more philosophical discussions as in the recent preprint of [2]), leading to many disparities between measures of complexity in the different models. As such, many of the existing results for quantum computation are difficult to compare with each other across the various fields and models.

Thus, it is the goal of this work to provide steps towards a framework to unify many of the existing models, and inspire future work to further explore ways to connect them. To do so, we begin in section (2) by introducing measures of complexity associated with models of quantum computation, and describing a selected subset of the major model classes relevant to our discussion in terms of their resources, including quantum circuits with discrete, continuous, or parameterized gate sets, as well as the controlled Schrödinger equation. Then, in section (3), we reformulate many of the existing equivalence results between these model classes, noting the distinction between bidirectional equivalence and unidirectional simulability. In particular, we note the importance of specifying the meaning of "equivalence."

Furthermore, as the prospects for quantum computing in the near future of the NISQ era will likely be done using hybrid quantum-classical algorithms implemented on parameterized quantum circuits (PQCs) [3], in section (4) we use insights gleaned from the reformulations in our framework to propose a notion of complexity for quantum operations in the model of PQCs, inspired in particular by bang-bang time-optimal control of the Schrödinger equation. We conclude in section (5) with some remarks on the possible implications of this new measure, as well as our hope that the aggregation and proposal presented within this framework could allow for future studies to successfully apply results and techniques *across* the existing models.

2 Models of Quantum Computation and Complexity

Quantum complexity is the natural extension of classical complexity theory to the quantum regime, where instead of a classical circuit or Turing machine, we have access to a *quantum computer*. For a full background on classical complexity theory, an introduction to quantum computation, and their intersection, we refer the reader to [4], [5], and [6] respectively. On this note, we also remark that the field of "quantum complexity theory" is and has been well established (see [7] for a comprehensive overview), but operates in a slightly different setting than those that we consider, since they tend to focus on a particular model (i.e. quantum Turing machine or quantum circuit) and examine complexity classes and algorithms relative to that choice of model. As such, one of the goals of this project is to unify various models such that the powerful tools and results from quantum complexity theory could be applied and interpreted within the context of the other models.

In general, a model of computation consists of a fixed setup and architecture, along with a set of one or more scalable resources. Defining the complexity of a model or a given problem involves measuring some function of the scalable resources. As such, problem definition is a key component of analyzing complexity. Determining whether a problem is solvable given a set of resource constraints is known as a decision problem (colloquially, a yes/no question), whereas finding the solution itself is known as a search problem, and finally an optimization problem involves minimizing a particular cost function subject to a set of constraints defined by the resources. Each of these instances involve intricacies that will be discussed in the following sections, particularly in the distinctions between classical and quantum resources, error scaling, and the universality of a particular model.

2.1 Quantum Circuit with Discrete Gate Set

The simplest model of quantum computation, after the purely-abstract quantum Turing machine, is a quantum circuit built from a discrete gate set. In the following definition, we adopt the traditional [5] "finite" restriction of a discrete set as opposed to allowing a countably infinite number of gates. Furthermore, a unitary gate U acts non-trivially on j out of n qubits (without loss of generality, assume it acts on the first j qubits) if j is the smallest value such that up to permutations of the qubits, U can be decomposed as $U_j \otimes \mathbb{1}_{2^{n-j}}$, where $\mathbb{1}_{2^{n-j}}$ is the identity operator on the remaining $n-j$ qubits.

Definition 2.1.1. *A discrete quantum gate set is a finite set of unitary gates defined by $\mathcal{G} = \{U_j\}_{1 \leq j \leq m}$, where each U_j acts non-trivially on any combination of c_j qubits.*

A discrete quantum circuit on n qubits is a sequence $\mathcal{S}_n$ of gates chosen from some $\mathcal{G}$, with $\mathcal{S}_n = \prod_{j=1}^{D(\mathcal{S}_n)} g_j$, with $g_j \in \mathcal{G}$. The scalable resources here are the gates in $\mathcal{G}$, where one key fact is that for a family of discrete quantum circuits $\{\mathcal{S}_n\}$ indexed by the number of qubits n, $\mathcal{G}$ remains constant and does not grow with n. Traditionally, the measure of complexity is some function of $\{g_j\}_{1 \leq j \leq D}$ with no concern for the order of applications, and usually the standard measure is simply the number of gates $D(\mathcal{S}_n)$ in the sequence. Another standard measure of complexity, denoted the depth $depth(\mathcal{S}_n)$ of the circuit, captures consecutive terms in $\mathcal{S}_n$ that act on disjoint sets of qubits into a single *layer*, and counts the total number of layers. Since for an n-qubit circuit at each layer at most n gates can be applied, we note that $D(\mathcal{S}_n) \leq n \cdot depth(\mathcal{S}_n)$. Therefore, $D(\mathcal{S}_n)$ is polynomially equivalent (in fact, by at most a linear factor in the number of qubits) to the depth of the circuit. One can also notice that even while fixing the depth of a circuit, one could vary the number of gates D by varying the number of qubits n, thereby presenting another important reason to distinguish between $D(\mathcal{S}_n)$ and $depth(\mathcal{S}_n)$.

A gate set $\mathcal{G}$ such that an arbitrary unitary operation can be constructed exactly is called exactly universal. Due to the uncountable number of possible quantum gates, it is well known that this is impossible for a discrete gate set $\mathcal{G}$; therefore, the goal of a discrete quantum circuit is to be able to *approximate* any unitary operation to arbitrary precision using a finite-length sequence of gates from $\mathcal{G}$. Such a gate set is called universal. In 2.2 below we briefly discuss the relationship between such concepts of exact and approximate universality.

As a result, the level of error ϵ defined by the operator norm $\|U_n - \mathcal{S}_n\|_{op}$ is also a measurable resource of the circuit that can scale with the number of qubits n. Usually, for a fixed number of qubits n, the measure of complexity is either defined to be the minimum $D(\mathcal{S}_n)$ as a function of ϵ, or the minimum $D(\mathcal{S}_n)$ for a fixed value of ϵ. For a family of circuits $\{\mathcal{S}_n\}$, the measure of complexity is the minimum $D(\mathcal{S}_n)$ as a function of both n and ϵ. In many cases, we want to guarantee that ϵ be at most constant in n, or even vanishing relative to n (denoted $o_n(1)$).

Precisely, the complexity $\mathfrak{C}[\{U_n\}]$ when one wants arbitrary-precision approximation is a *function* of n for a family of unitary gates $\{U_n\}$:

$$\mathfrak{C}[\{U_n\}] := \sup_{\epsilon > 0} \left(\inf_{\mathcal{S}_n \ s.t. \ \|U_n - \mathcal{S}_n\|_{op} \leq \epsilon} \Big(D(\mathcal{S}_n) \Big) \right) \tag{1}$$

Here, the inner infimum returns a function of n, and so afterwards we take the outer supremum

over functions (usually defined by the standard hierarchy of highest-order terms). On the other hand, for a particular (possibly functional) value of ϵ as ϵ_n, we can write:

$$\mathfrak{C}[\{U_n\}, \{\epsilon_n\}] := \inf_{\mathcal{S}_n \ s.t. \ \|U_n - \mathcal{S}_n\|_{op} \leq \epsilon_n} \Big(D(\mathcal{S}_n) \Big) \tag{2}$$

2.2 Quantum Circuit with Continuous Gate Set

A quantum circuit built instead from an uncountable and/or continuous gate set is a generalization of the discrete gate set:

Definition 2.2.1. *A continuous quantum gate set is a finite set of families of unitary gates defined by $\mathcal{G} = \{F_j\}_{1 \leq j \leq m}$, where each F_j is a (possibly uncountable) family of gates that act non-trivially on any combination of c_j qubits.*

Some common choices for F_j are S_k (the set of all unitary gates acting non-trivially on k qubits, as distinguished from $\mathcal{S}_k$, a particular circuit on k qubits), or arbitrary rotations $\{e^{-i\alpha H}\}_{\alpha \in \mathbb{R}}$ along a particular axis defined by a self-adjoint operator H. Unlike discrete quantum gate sets, continuous quantum gate sets can be exactly universal. As a result, the level of error ϵ_n upper bounding $\|U_n - \mathcal{S}_n\|_{op}$ is sometimes fixed to be 0 instead of some positive value.

Furthermore, in many cases each family of gates $F_j \in \mathcal{G}$ has a different measure of complexity, as for example in the case of the exactly universal gate set $\mathcal{G} = S_1 \cup \{\text{CNOT}\}$ proven by Barenco et al. [8], where the measure of complexity is simply the number of CNOTs applied in the sequence (and where CNOT is a particular 2-qubit gate [5]).

Thus, as opposed to discrete quantum circuits, continuous quantum circuits yield many choices in complexity measures, for example by choosing between zero and constantly-bounded error, and in the weighting of gates from each family F_j. At this juncture, we note that although it has been shown [9, 10] that a particular gate set $\mathcal{G}$ is exactly universal if and only if it is also approximately universal (in the sense that it can approximate any unitary up to arbitrary precision), this does not necessarily mean that the *complexities* associated with them are equivalent. Indeed, the number of gates from $\mathcal{G}$ required to exactly implement some U_n is strictly higher than the number of gates required to implement U_n up to some precision ϵ_n - in fact, it is yet unclear whether the two measures are even polynomially equivalent with respect to either n or ϵ_n. As such, it is important to explore these intricacies when discussions move beyond simple *universality* and into *measuring* the implementations.

2.2.1 Parameterized Quantum Circuit

A particular instance of a quantum circuit with a continuous gate set is known as a *parameterized quantum circuit* (PQC). Here, each family of unitary gates in the gate set is defined by a particular Hamiltonian and either zero or one parameters. Formally:

Definition 2.2.2. *A parameterized quantum gate set is a finite set of families of unitary gates defined by* $\mathcal{G} = \{e^{-i\alpha_j H_{j,1}}, e^{-iH_{j,2}}\}_{1\leq j\leq m}$*, where* $\alpha_j \in \mathbb{R}$ *are continuous parameters, and each* $H_{j,1}$ $(H_{j,2})$ *are Hermitian operators acting non-trivially on any combination of* $c_{j,1}$ $(c_{j,2})$ *qubits.*

Here, the unitary corresponding to $H_{j,1}$ is parameterized by a real parameter α_j, whereas the unitaray corresponding to $H_{j,2}$ is fixed. In practice, the set $\mathcal{G}$ is in fact ordered, and for a fixed number of qubits n the locations $c_{j,1}, c_{j,2}$ of the operators $H_{j,1}, H_{j,2}$ are fixed as well. This defines an *ansatz* for a particular parameterized quantum circuit, in the sense that the architecture is predetermined with depth $2m$ and parameter count m (see [11] for a review of PQCs in the context of variational quantum algorithms). Therefore, the "traditional" resource in a parameterized quantum circuit is the number of classical parameters m, similarly to the standard choice for discrete and continuous quantum circuits. There are also other measures such as "effective dimension" [12] and other neural-network based measures of complexity or expressivity [13].

Similarly as to how a continuous quantum circuit is a strict generalization of the discrete quantum circuit, a parameterized quantum circuit can easily be seen as a particularization of a continuous quantum circuit. In this sense, there are constraints on both the sets of allowable/parameterized gates, as well as on the *architecture* of the circuit, which we briefly discuss in section (4) especially in the connection to bang-bang time-optimal quantum control.

2.3 Controlled Schrödinger Equation

On the other hand, the model of computation usually referred to as "quantum control" is fundamentally different from the aforementioned families of quantum circuits, as it operates in continuous rather than discrete time. In fact, perhaps a better moniker for this model is the "controlled Schrödinger equation." This is based on the fact that the time evolution of a closed

quantum system is governed by the (normalized) time-dependent Schrödinger equation

$$\frac{dU(t)}{dt} = -iH(t)U(t), \quad U(0) = \mathbb{1} \tag{3}$$

$$U(T) = \mathcal{T}\left[\exp\{-i\int_0^T H(t)\ dt\}\right] \tag{4}$$

where $\mathcal{T}$ denotes time-ordering. As such, altering $H(t)$ for $0 \leq t \leq T$ allows for the construction of a unitary gate $U(T)$. In this sense, one is "controlling" the Hamiltonian that governs the evolution of the system.

This model is very broad, allowing for many types of constraints on $H(t)$. One common constraint is to expand $H(t)$ in some basis (such as the generalized Pauli basis) and bound and/or penalize the individual coefficients in the basis expansion (such as only allowing for $H(t)$ to consist of up to k-body operators for some fixed k). Another common formulation in quantum control is to decompose $H(t)$ into a *drift* Hamiltonian and a set of *control* Hamiltonians, defining a bilinear control system:

$$H(t) = H_0 + \sum_{k=1}^{m} f_k(t)H_k \tag{5}$$

where each $H_{0\leq k\leq m}$ are fixed and $f_k(t) \in \mathbb{R}$. In this work, we will refer to "quantum control" as defining the class of controlled Schrödinger equations of this form. Some common constraints are to bound $|f_k(t)| \leq c$ for some constant c, or to restrict the control function $\{f_k(t)\} \in \mathcal{C}^1(\mathbb{R})$ to be continuous, or differentiable/smooth/bang-bang/square-integrable/etc.

Under any set of constraints, many natural measures of complexity arise. The simplest measure is the time T itself, denoted the *minimum time* (although the formal term would be an *infimizing* time, as we can see in the definition):

$$T^* = \inf_{U(T)=U} T \tag{6}$$

where the infimum is taken over all $U(t)$ with its generator $H(t)$ satisfying the given constraints. As in the case of continuous quantum circuits, the model of the controlled Schrödinger equation allows for exact universality/controllability, and as such the level of error can be either set to zero or bounded by a constant, and other proposed measures will be discussed in the ensuing sections. While in this work we focus on a particular selection of results, the field of quantum control extends far beyond what we discuss. For some recent overviews we refer to [14, 15, 16].

3 Notions of Equivalence

Before we discuss notions of equivalence between these various models, we first make a brief note on the difference between and separation of "classical" and "quantum" resources in any quantum model of computation. Even though any classical computation can be simulated by a quantum computer, we normally separate the two components.

In particular, the classical difficulty of "finding" the optimal implementation of an operation under any particular model (for example, finding the optimal gate sequence or the right controls/parameters for an evolution) is usually separate from the aforementioned measures of quantum complexity of implementing that operation, such as the number of gates/parameters or the total time. In this work we focus mainly on the second part, although in discussing the proposals for parameterized quantum circuits in section (4) we will refer to connections to the classical complexity.

As such, having defined the relevant quantum models of computation, we now reformulate many of the existing equivalence results, both between and within each class. In order to talk about equivalence, one needs to fix two models of computation and then equate the two varying/scaling resources (such as the number of gates from a universal gate set compared to the total time in a time-optimal evolution under a controlled Schrödinger equation). When one class is thought to be more powerful than another, "equivalence" necessarily entails constraining one class, boosting the other class, or constructing particular metrics. Hearkening back to section (2.2), we note the important difference between "equivalence" in reachability/implementability/universality, and "equivalence" in terms of a particular measure of resources.

Because there are in fact *many* "quantum" resources at play, resource theories have been suggested to equate many of them, such as entanglement (see for examaple [17]). However, these existing theories are not the same as what we are attempting to do, since they don't consider a broader class of models and instead focus on the resources present within a single model.

Similarly, the famous Solovay-Kitaev theorem [18] was one of the first presentations of model equivalence for quantum circuits with discrete gate sets, which informally states that *all* universal discrete gate sets are polynomially equivalent up to polylogarithmic error. Formally, one can write, where a gate set $\mathcal{G}$ *contains its own inverses* if for all $g \in \mathcal{G}$, we also have $g^{-1} \in \mathcal{G}$:

Theorem 3.1. *(Solovay-Kitaev 1997 [18], refined in [6, 19]) An n-qubit quantum circuit constructed by a sequence of m quantum gates from a discrete gate set $\mathcal{G}_1$ that contains its own*

inverses can be approximated to error ϵ in operator norm by a quantum circuit of $\mathcal{O}\Big(m \log^4(\frac{m}{\epsilon})\Big)$ gates from any discrete gate set $\mathcal{G}_2$ that also contains its own inverses. Setting $\epsilon = \mathcal{O}(\frac{1}{n})$ yields a simulation length of $\mathcal{O}\Big(m \log^4(mn)\Big)$.

In this sense, any discrete gate set that contains its own inverses can *polynomially simulate* any other such discrete gate set, and more importantly, the simulation is done in such a way that is only *polylogarithmic* in the quantity $\frac{1}{\epsilon}$ associated with the error. However, an important caveat is that this initial sequence of m gates from $\mathcal{G}_1$ may itself be exponential in n - as such, while this result states that a discrete gate set $\mathcal{G}_2$ could polynomially simulate $\mathcal{G}_1$, it remains difficult to characterize and/or find the initial sequence using $\mathcal{G}_1$. Furthermore, as we will see, extending even the polynomial equivalence result to continuous gate sets and further into continuous-time models is significantly more challenging. In the following sections we reformulate a few key attempts at doing so.

3.1 Nielsen

3.1.1 Circuit Depth is Lower Bounded by a "Geodesic Distance"

We reformulate the relevant parts of the result presented in the article "A geometric approach to quantum circuit lower bounds," published in 2005 by Nielsen [20].

Consider a family of unitaries $\{U_n\}$ that we want to implement to error ϵ that is constant and independent of n. Let $\mathcal{G} = S_1 \cup S_2$ be a continuous quantum gate set defined by all one- and two-qubit unitaries. We define the complexity of a particular sequence $\mathcal{S}_n$ with elements in $\mathcal{G}$ to simply be $|\mathcal{S}_n|$, or the total number of one- and two-qubit gates. Thus, $\mathfrak{C}_{\mathcal{G}}[U_n]$ is defined to be the infimum of $|\mathcal{S}_n|$ over all sequences $\mathcal{S}_n$ with $\|U_n - \mathcal{S}_n\|_{op} \leq \epsilon$.

Effectively, we show that a lower bound for $\mathfrak{C}_{\mathcal{G}}[U_n]$ is given by the length of the minimal geodesic between U_n and the identity $\mathbb{1}_n$, where length is defined by a suitable Finsler metric on the manifold $SU(2^n)$. To do so, define an *instantaneous cost function* $F(H(t))$ that measures the cost of applying a particular Hamiltonian $H(t)$, with the condition that F is derived from a right-invariant local metric on $SU(2^n)$ (although we treat it as a norm on $su(2^n)$ - for further information on the precise properties of F, see section IIB in [20]). For a particular $H(t)$ that generates $U(t)$ with $U(T) = U_n$, define the following complexity $\mathfrak{C}_{FM}$ (for Finsler metric) as

measured across the entire evolution $0 \leq t \leq T$ (denoted $U(t \leq T)$):

$$\mathfrak{C}_{FM}[U(t \leq T)] := \int_0^T F(H(t))\ dt \tag{7}$$

$$\mathfrak{C}_{FM}[U_n] := \inf_{T>0\ s.t.\ U(T)=U_n} \mathfrak{C}_{FM}[U(t \leq T)] \tag{8}$$

Now, we show the following theorem (with the n-subscript dropped for clarity):

Theorem 3.2. *If a Finsler metric F satisfies $F(H(t)) \leq 1$ for $H(t)$ corresponding to the generator for any one- or two-qubit unitary, then we have that $\mathfrak{C}_{FM}[U] \leq \mathfrak{C}_{\mathcal{G}}[U]$. As such, the minimum number of gates required to synthesize U is lower bounded by the metric given by $\mathfrak{C}_{FM}[U]$.*

Proof. (we use the notation in [20]) Suppose that $U_1 = e^{-iH_1}, \ldots, U_{\mathfrak{C}_{\mathcal{G}}[U]} = e^{-iH_{\mathfrak{C}_{\mathcal{G}}[U]}}$ defines a minimal sequence $\mathcal{S}$ to implement U from the gate set $\mathcal{G}$ of one- and two-qubit gates. We define the following curve $H(t)$ as in equation 13 (Theorem 1, page 9) of [20]:

$$\frac{1}{\mathfrak{C}_{\mathcal{G}}[U]} H(t) = \left\{ \begin{array}{ll} H_1, & \text{for } 0 \leq t \leq \frac{1}{\mathfrak{C}_{\mathcal{G}}[U]} \\ H_2, & \text{for } \frac{1}{\mathfrak{C}_{\mathcal{G}}[U]} \leq t \leq \frac{2}{\mathfrak{C}_{\mathcal{G}}[U]} \\ \ldots & \ldots \\ H_{\mathfrak{C}_{\mathcal{G}}[U]}, & \text{for } 1 - \frac{1}{\mathfrak{C}_{\mathcal{G}}[U]} \leq t \leq 1 \end{array} \right\}$$

Since this $H(t)$ is not smooth, so to make it smooth we can regularize it using a function $r(t)$ such that $r(t)H(t)$ also generates $U(t)$. Choose an $r(t)$ such that $r(t) \geq 0$, $r(t) = 0$ for $t = \frac{k}{\mathfrak{C}_{\mathcal{G}}[U]}$ for all relevant integers k, and that for any such integer k we also have:

$$\int_{\frac{k}{\mathfrak{C}_{\mathcal{G}}[U]}}^{\frac{k+1}{\mathfrak{C}_{\mathcal{G}}[U]}} r(t)\ dt = \frac{1}{\mathfrak{C}_{\mathcal{G}}[U]} \tag{9}$$

Such $r(t)$ is easily constructed, as we can notice for example the function $r(t) = 2\sin^2(t\pi \cdot \mathfrak{C}_{\mathcal{G}}[U])$

works. Now, since $r(t)H(t)$ also generates $U(t)$ for $0 \leq t \leq 1$, we obtain:

$$\begin{aligned}
\mathfrak{C}_{FM}[U] &\leq \mathfrak{C}_{FM}[U(t \leq 1)] && (10)\\
&= \int_0^1 F(r(t)H(t))\ dt && (11)\\
&= \int_0^1 r(t) \cdot F(H(t))\ dt && (12)\\
&= \int_0^1 r(t) \cdot F(\mathfrak{C}_{\mathcal{G}}[U] \cdot \frac{1}{\mathfrak{C}_{\mathcal{G}}[U]} H(t))\ dt && (13)\\
&= \int_0^1 r(t) \cdot \mathfrak{C}_{\mathcal{G}}[U] \cdot F(\frac{1}{\mathfrak{C}_{\mathcal{G}}[U]} H(t))\ dt && (14)
\end{aligned}$$

Since each term in $\frac{1}{\mathfrak{C}_{\mathcal{G}}[U]}H(t)$ is the generator for a one- or two- qubit unitary by construction, we know from the inequality given by F that $F(\frac{1}{\mathfrak{C}_{\mathcal{G}}[U]}H(t)) \leq 1$; this yields:

$$\begin{aligned}
\mathfrak{C}_{FM}[U] &\leq \int_0^1 r(t) \cdot \mathfrak{C}_{\mathcal{G}}[U]\ dt && (15)\\
&= \mathfrak{C}_{\mathcal{G}}[U] \cdot \int_0^1 r(t)\ dt && (16)\\
&= \mathfrak{C}_{\mathcal{G}}[U] \sum_{k=0}^{\mathfrak{C}_{\mathcal{G}}[U]-1} \Big(\int_{\frac{k}{\mathfrak{C}_{\mathcal{G}}[U]}}^{\frac{k+1}{\mathfrak{C}_{\mathcal{G}}[U]}} r(t)\ dt \Big) && (17)\\
&= \mathfrak{C}_{\mathcal{G}}[U] \sum_{k=0}^{\mathfrak{C}_{\mathcal{G}}[U]-1} \frac{1}{\mathfrak{C}_{\mathcal{G}}[U]} && (18)\\
&= \sum_{k=0}^{\mathfrak{C}_{\mathcal{G}}[U]-1} 1 && (19)\\
&= \mathfrak{C}_{\mathcal{G}}[U] && (20)
\end{aligned}$$

Therefore, $\mathfrak{C}_{FM}[U] \leq \mathfrak{C}_{\mathcal{G}}[U]$, as desired. □

While this is an interesting proof, we notice the following insight when attempting to connect the result to the quantum control model:

Remark 3.1. *In order to apply Theorem (3.2) to the quantum control model, one effectively requires that the control field amplitudes be bounded. Furthermore, since the bound is constructed by simulating each gate in the sequence, it does not provide any insight when considered as an upper bound on the minimum time (both with and without drift).*

3.1.2 Circuit Depth is Polynomial in a "Geodesic Distance"

We reformulate the result presented in the article "Quantum Computation as Geometry," published in 2006 by Nielsen et al. in *Science* [21], which is similar in setting to the previous equivalence result published in 2005 [20] by Nielsen, described above.

As before, consider a family of unitaries $\{U_n\}$ that we want to implement to error ϵ that is constant and independent of n. Let $\mathcal{G} = S_1 \cup S_2$ be a continuous quantum gate set defined by all one- and two-qubit unitaries. Again, we define the complexity of a particular sequence $\mathcal{S}_n$ with elements in $\mathcal{G}$ to simply be $|\mathcal{S}_n|$, or the total number of one- and two-qubit gates. Thus, $\mathfrak{C}_{\mathcal{G}}[U_n]$ is defined to be the infimum of $|\mathcal{S}_n|$ over all sequences $\{\mathcal{S}_n\}$ with $\|U_n - \mathcal{S}_n\|_{op} \leq \epsilon$.

On the other hand, implementing U_n through a controlled Schrödinger equation yields a Hamiltonian $H(t)$ such that $U(T) = U_n$ for some T. Then, one can expand $H(t)$ in the Pauli basis as follows (with P_n being the set of Pauli operators acting nontrivially on n qubits):

$$H(t) = \sum_{H_j \in P_1 \cup P_2} f_j(t) H_j + \sum_{H_k \in P_3 \cup \cdots \cup P_n} g_k(t) H_k \tag{21}$$

where the terms are split according to those corresponding to one- or two-qubit operators, and those corresponding to many-body operators. This yields the following *instantaneous cost* of applying a particular Hamiltonian $H(t)$:

$$F(H(t)) = \sqrt{\left[\sum_{H_j \in P_1 \cup P_2} f_j^2(t)\right] + p^2 \left[\sum_{H_k \in P_3 \cup \cdots \cup P_n} g_k^2\right]} \tag{22}$$

where p is a penalty paid for applying many-body terms, which is later chosen to be large in order to suppress those terms when wanting to minimize $F(H(t))$. One can notice that F is effectively a weighted 2-norm on the coefficients of $H(t)$. Thus, one can define the complexity $\mathfrak{C}_{FM}$:

$$\mathfrak{C}_{FM}[U(t \leq T)] := \int_0^T F(H(t))\ dt \tag{23}$$

$$\mathfrak{C}_{FM}[U_n] := \inf_{T>0 \ \text{s.t.}\ U(T)=U_n} \mathfrak{C}_{FM}[U(t \leq T)] \tag{24}$$

In Nielsen's article, $H(t)$ is rescaled such that $F(H(t)) = 1$ for all t and as such $\mathfrak{C}_{FM}[U(t \leq T)] = T$; this is done not only to intuitively tie in the time T to the measure of complexity, but mainly to connect it to the particular choice of F in the previous equivalence result above. This

will also be discussed later, but a noticeably caveat to point out is that this direction does not work if there is a nonzero drift component, since $H(t)$ cannot be arbitrarily rescaled.

The first step in the proof is to project the Hamiltonian $H(t)$ onto one that only contains one- and two-body terms, and justify doing so by choosing the penalty term p appropriately. Here, we define $H_\pi(t)$ to simply be the component of $H(t)$ corresponding to the one- and two-body terms:

$$H_\pi(t) = \sum_{H_j \in P_1 \cup P_2} f_j(t) H_j \tag{25}$$

Then, defining $U_\pi(t)$ to be the unitary generated by $H_\pi(t)$, we can now show that:

Lemma 3.3.

$$\|U_n - U_\pi(T)\|_{op} \leq \frac{2^n \cdot \mathfrak{C}_{FM}[U_n]}{p} \tag{26}$$

Proof. Choose $U(t \leq T)$ such that $U(T) = U_n$ and such that $\mathfrak{C}_{FM}[U(t \leq T)] = \mathfrak{C}_{FM}[U_n]$. We have:

$$\begin{aligned}
\mathfrak{C}_{FM}[U_n] &= \mathfrak{C}_{FM}[U(t \leq T)] \\
&= \int_0^T F(H(t))\ dt \\
&= \int_0^T \sqrt{\left[\sum_{H_j \in P_1 \cup P_2} f_j^2(t)\right] + p^2 \left[\sum_{H_k \in P_3 \cup \cdots \cup P_n} g_k^2\right]}\ dt \\
&\geq \int_0^T \sqrt{p^2 \left[\sum_{H_k \in P_3 \cup \cdots \cup P_n} g_k^2\right]}\ dt \\
&= p \cdot \int_0^T \sqrt{\sum_{H_k \in P_3 \cup \cdots \cup P_n} g_k^2}\ dt \\
&= p \cdot \int_0^T \|H(t) - H_\pi(t)\|_2\ dt
\end{aligned}$$

By Cauchy-Schwarz, we know that $\|H\|_{op} \leq 2^n \|H\|_2$ for any H, and for any unitarily invariant

norm $\|\cdot\|$ we know that $\|U_1(T) - U_2(T)\| \leq \int_0^T \|H_1(t) - H_2(t)\|\ dt$. Thus:

$$\begin{aligned}
\mathfrak{C}_{FM}[U_n] &\geq p \cdot \int_0^T \|H(t) - H_\pi(t)\|_2\ dt \\
&\geq \frac{p}{2^n} \cdot \int_0^T \|H(t) - H_\pi(t)\|_{op}\ dt \\
&\geq \frac{p}{2^n} \cdot \|U(T) - U_\pi(T)\| \\
&= \frac{p}{2^n} \cdot \|U_n - U_\pi(T)\|
\end{aligned}$$

Rearranging immediately yields the Lemma, as desired. □

Thus, choosing $p = 4^n$ yields $\|U_n - U_\pi(T)\|_{op} \leq \frac{\mathfrak{C}_{FM}[U_n]}{2^n}$. Now, we discretize $H_\pi(t)$ into small intervals of length Δ, where each interval can be simulated by a constant mean Hamiltonian denoted $\overline{H}_{\pi,\Delta}$. This yields the second Lemma in the proof:

Lemma 3.4. *Let $H(t \leq \Delta)$ generate a unitary U, such that $\|H(t)\|_{op} \leq c$ for all $0 \leq t \leq \Delta$. Then, we have:*

$$\overline{H} = \frac{1}{\Delta} \int_0^\Delta H(t)\ dt \tag{27}$$

$$\|U - e^{-i\Delta\overline{H}}\|_{op} \leq 2(e^{c\Delta} - 1 - c\Delta) = \mathcal{O}(c^2\Delta^2) \tag{28}$$

Proof. We can recall the Dyson series for $U = U(\Delta)$ as follows:

$$U(\Delta) = \sum_{m=0}^{\infty} (-i)^m \int_0^\Delta dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m\ H(t_1)H(t_2)\cdots H(t_m) \tag{29}$$

We can then write (where each norm is implicitly the operator norm):

$$\|U - e^{-i\Delta\overline{H}}\| = \|\sum_{m=0}^{\infty}(-i)^m \int_0^{\Delta} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \; H(t_1)H(t_2)\cdots H(t_m) \tag{30}$$

$$- \sum_{m=0}^{\infty} \frac{(-i\Delta\overline{H})^m}{m!}\| \tag{31}$$

$$= \|\mathbb{1} - \mathbb{1} + (-i)\int_0^{\Delta} H(t_1)\; dt_1 - (-i\Delta\overline{H}) \tag{32}$$

$$+ \sum_{m=2}^{\infty}\Bigg[(-i)^m \int_0^{\Delta} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \; H(t_1)H(t_2)\cdots H(t_m) \tag{33}$$

$$- \frac{(-i\Delta\overline{H})^m}{m!}\Bigg]\| \tag{34}$$

$$= \|\sum_{m=2}^{\infty}\Bigg[(-i)^m \int_0^{\Delta} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \; H(t_1)H(t_2)\cdots H(t_m) \tag{35}$$

$$- \frac{(-i\Delta\overline{H})^m}{m!}\Bigg]\| \tag{36}$$

$$\leq \sum_{m=2}^{\infty}\Bigg[\int_0^{\Delta} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \; \|H(t_1)H(t_2)\cdots H(t_m)\| \tag{37}$$

$$+ \frac{\|(-i\Delta\overline{H})^m\|}{m!}\Bigg] \tag{38}$$

$$\leq \sum_{m=2}^{\infty}\Bigg[c^m \cdot \frac{\Delta^m}{m!} + \frac{\Delta^m c^m}{m!}\Bigg] \tag{39}$$

$$= 2\sum_{m=2}^{\infty}\frac{c^m\Delta^m}{m!} = 2(e^{c\Delta} - 1 - c\Delta) \tag{40}$$

This completes the proof, since we now have $\|U - e^{-i\Delta\overline{H}}\| \leq c^2\Delta^2 + \mathcal{O}(c^3\Delta^3) = \mathcal{O}(c^2\Delta^2)$. □

Now, from page 56 of "Matrix Computations" by Golub and Van Loan (1996; [22]), we know that:

$$\|H_\pi(t)\|_{op} \leq \sqrt{rank(H_\pi(t))} \cdot \|H_\pi(t)\|_F \tag{41}$$

$$\leq \sqrt{16n^2} \cdot F(H_\pi(t)) \tag{42}$$

$$= 4n \cdot F(H_\pi(t)) \tag{43}$$

where the rank is $\leq 16n^2$ since $H_\pi(t)$ contains at most one- and two-body terms, of which there are at most $16n^2$. For the purposes of our setting, we know that $\|H_\pi(t)\|_{op} \leq 4n \cdot F(H_\pi(t))$, and since we rescale such that $F(H_\pi(t)) \leq F(H(t)) = 1$ (again, not valid in the case of nonzero drift), we obtain that $\|H_\pi(t)\|_{op} \leq 4n$, so applying Lemma 3.3 to the unitary U generated by

$H_\pi(t \leq \Delta)$ yields $\|U - e^{-i\Delta\overline{H}}\| = \mathcal{O}(n^2\Delta^2)$.

Remark 3.2. *It is also important to note that this inequality is precisely the issue in the case with non-zero drift, as $F(H_\pi(t))$ cannot be bounded by rescaling $H_\pi(t)$. This section shows that Nielsen's result cannot hold for cases with unbounded control, and in the case of bounded control is valid when the norm of H_0 can be bounded by a polynomial in n.*

Many minimum-time results from the quantum control model derived using Nielsen's inequality involve such bounds on $\|H_0\|_{op}$, which could lead to some interesting explorations.

From this, we can show the third and final Lemma in the proof:

Lemma 3.5. *An n-qubit constant Hamiltonian $\overline{H}_{\pi,\Delta}$, which only contains one- and two-body terms, can be simulated using a sequence of $\mathcal{O}(\frac{n^2}{\Delta})$ one- and two-qubit gates.*

Proof. Divide the interval $[0, \Delta]$ into $N = \frac{1}{\Delta}$ steps of size Δ^2, and define

$$U_{\Delta^2,k} = \prod_{H_j \in P_1 \cup P_2} e^{-i\alpha_{j,k}H_j}$$

Here, each term $e^{-i\alpha_{j,k}H_j}$ is a one- or two-qubit gate in the continuous gate set $\mathcal{G}$, and it is standard to show that since $|P_1 \cup P_2| = \mathcal{O}(n^2)$:

$$\|e^{-i\overline{H}_{\pi,\Delta}\Delta^2} - U_{\Delta^2,k}\|_{op} = \mathcal{O}(n^4\Delta^4)$$

Then, using repeated triangle inequality and unitary invariance, we obtain:

$$\begin{aligned}
\|e^{-i\overline{H}_{\pi,\Delta}\Delta} - \prod_{k=1}^{N} U_{\Delta^2,k}\|_{op} &= \|\prod_{k=1}^{N} e^{-i\overline{H}_{\pi,\Delta}\frac{\Delta}{N}} - \prod_{k=1}^{N} U_{\Delta^2,k}\|_{op} \\
&= \|\prod_{k=1}^{N} e^{-i\overline{H}_{\pi,\Delta}\Delta^2} - \prod_{k=1}^{N} U_{\Delta^2,k}\|_{op} \\
&\leq \sum_{k=1}^{N} \|e^{-i\overline{H}_{\pi,\Delta}\Delta^2} - U_{\Delta^2,k}\|_{op} \\
&\leq N \cdot \mathcal{O}(n^4\Delta^4) \\
&= \mathcal{O}(n^4 \cdot \Delta^3)
\end{aligned}$$

The number of gates used here is $\mathcal{O}(N \cdot n^2) = \mathcal{O}(\frac{n^2}{\Delta})$. □

Putting these together yields the following theorem:

Theorem 3.6. *Let $\mathcal{G}$ be the gate set consisting of all one- and two- qubit gates, U_n be an n-qubit unitary, and $\mathfrak{C}_{\mathcal{G}}, \mathfrak{C}_{FM}$ be the complexities defined above. Then:*

$$\mathfrak{C}_{\mathcal{G}}[U_n] = \mathcal{O}(n^6 \cdot \mathfrak{C}_{FM}[U_n]^3) \tag{44}$$

In other words, the minimum number of gates from $\mathcal{G}$ required to implement U_n is polynomial in both the number of qubits n and the constructed complexity $\mathfrak{C}_{FM}$. (Note: the main body of Nielsen et al. 2006 [21] (page 1135) contains the full proof, although the notation here is rewritten for our context.)

This shows that the natural measure of complexity in the continuous quantum circuit defined by one- and two-qubit gates is polynomial in a geodesic distance defined in the controlled Schrödinger equation. Unfortunately, it is nontrivial to extend this to the case where there is drift.

Now, we briefly consider the special case where $H(t)$ only has one- and two-body terms. Here, the parameter p used in the construction of $H_\pi(t)$ from $H(t)$ is not necessary, since the original Hamiltonian $H(t)$ already satisfies the properties of only containing one- and two-body terms; in fact, the proof of Theorem 3.4 would be equivalent even when only considering this special case for $H(t)$, as the bottleneck in the gate count does not depend strongly on the $H(t) \to H_\pi(t)$ relaxation. This yields the following:

Corollary 3.6.1. *Let an n-qubit Hamiltonian $H(t)$ generate a unitary U_n at time $t = T$ such that the quantity $\int_0^T \|H(t)\|_2 \, dt$ is minimal over all such Hamiltonians $H(t)$ generating U_n that consist only of one- and two-qubit terms for all times $0 \leq t \leq T$. Then, the minimum number of one- and two- qubit gates required to synthesize U_n, denoted $\mathfrak{C}_{\mathcal{G}}[U_n]$, scales as*

$$\mathfrak{C}_{\mathcal{G}}[U_n] = \mathcal{O}\left(n^6 \cdot \left(\int_0^T \|H(t)\|_2 \, dt\right)^3\right) \tag{45}$$

One example of such $H(t)$ could involve a non-zero drift H_0 that only contains one- and two-body terms, and control Hamiltonians that similarly only contain one- and two-body terms. One standard setting is an Ising Hamiltonian H_0 with full local control. Here, we see that $F(H(t))$ scales at most quadratically in n provided the control fields are bounded, in which case Nielsen's result can hold. The case of unbounded control fields requires further exploration. Furthermore:

Remark 3.3. *A bang-bang sequence involving a 2-local drift Hamiltonian H_0 and full local controls could also utilize Nielsen's result. Given a minimum-time bang-bang sequence, each single-qubit gate is already included as an entry in the gate set, so the only relevant portions are of the form $e^{-iH_0 t}$ for some value of t. Since $\|tH_0\| = t\|H_0\|$, provided the minimum-time is polynomial in n, Nielsen's result immediately holds even without the requirement of bounded control fields.*

This observation is therefore one reason to justify our focus on looking at the conditions where a bang-bang protocol can be optimal.

3.1.3 Comments on Nielsen's Results

The first natural question to ask is whether or not $\mathfrak{C}_{FM}[U_n]$ from section (3.1.1) or $\mathfrak{C}_{FM}[U_n]$ from section (3.1.2) can be connected to the physical time T. In the case without drift, this question is trivial since the physical time T itself does not matter (due to arbitrary rescaling), and Nielsen does this already by setting $F(H(t)) = 1$ so that $\mathfrak{C}_{FM}[U(t \leq T)] = T$ in the "upper bound" case. For the case with drift, however, the lack of arbitrary rescaling makes the time T not be the quantity that minimizes $\mathfrak{C}_{FM}[U(t \leq T)]$. Furthermore, the "lower bound" effectively gives an upper bound on the minimum time, but only for a particular control system - in effect, the bound arises by simulating each gate in sequence, which is a trivial construction and therefore not particularly useful.

Another important viewpoint is that both of Nielsen's articles [20] and [21] aim to provide upper and lower bounds on the minimal circuit depth/complexity, not from the perspective of quantum control or minimum times. In this sense, since as mentioned above the lower bound on circuit complexity is effectively irrelevant to minimum-time considerations, only the upper bound is potentially useful. However, as also mentioned above, the cases with drift and/or unbounded controls unfortunately do not fit the main criteria of the results (arbitrary rescaling and bounded norm). It is therefore the opinion of this work that "polynomial equivalence" between minimum time and minimum circuit depth has yet to be proven for the case with non-zero drift. We can further make the following remark:

Remark 3.4. *We can re-organize the relations provided by Nielsen in the two papers [20, 21] as the following "hierarchy" of inequalities:*

1. *Given a Finsler metric F, the geodesic distance $\mathfrak{C}_{FM}[U_n]$ from the identity $\mathbb{1}$ to an n-qubit goal unitary U_n constructed with respect to F*

$\leq$

2. *The value of $\mathfrak{C}_{FM}[U(t \leq T)]$ for a particular unitary $U(t \leq T)$ satisfying $U(T) = U_n$, where $U(t \leq T)$ is constructed by simulating each gate in a minimal-length gate sequence of one- and two- qubit gates from $\mathcal{G}$ that creates U_n*

$\leq$ *[20]*

3. *The length of the minimum gate sequence consisting of one- and two-qubit gates required to synthesize U_n*

$\leq$

4. *The length of a particular gate sequence of one- and two- qubit gates that synthesizes U_n by approximating geodesic time-evolution $U(t \leq T)$*

$\leq$ *[21]*

5. $\mathcal{O}\left(n^6 \cdot \mathfrak{C}_{FM}^3[U_n]\right)$, *where $\mathfrak{C}_{FM}[U_n]$ is the geodesic distance from (1)*

One can observe that indeed, the value that would be associated with a minimum time T, namely the geodesic distance $\mathfrak{C}_{FM}[U_n]$, only appears on the outer ends of the inequalities, thereby not allowing for any particularly helpful results from a practical standpoint.

3.2 Sussmann, Khaneja, etc.: The Case for Bang-Bang Control

Here we highlight results from the controlled Schrödinger equation model that relate in particular to the case of bang-bang time-optimal control.

3.2.1 Jurdjevic/Sussmann 1972

We reformulate key results presented in the article "Control Systems on Lie Groups," published in 1972 by Jurdjevic and Sussmann [9]. On a similar note, we mention that a related paper from later in 1972 by Sussmann, entitled "The Bang-Bang Problem for Certain Control Systems in $GL(n, \mathbb{R})$" [23] uses a different definition of "bang-bang" from our setting, which we do not focus on here.

The setting for the controlled Schrödinger equation is as defined above, with the bilinear control system $H(t) = H_0 + \sum_{k=1}^{m} f_k(t) H_k$. Let the subalgebra $\mathbf{L}$ be generated by $-iX_0, \ldots, -iX_m$, with the subalgebra L generated by the controls $-iX_1, \ldots, -iX_m$, and corresponding connected Lie subgroups $\mathbf{S}, S$ respectively. Let the reachable set $\mathcal{R}_U$ be the set reachable from U in finite

time, and for particular T the reachable set is $\mathcal{R}_U(T)$. Let F_u be the class of all locally bounded and measurable functions defined on the interval $[0, \infty)$ and having values in $\mathbb{R}^m$, and F_b be the class of piecewise-constant functions defined on $[0, \infty)$ with components taking values in $\{-1, 1\}$. Then, we have:

Theorem 3.7. *(Theorem 5.1 in [9]) For a driftless bilinear control system with $H_0 = \mathbf{0}$ and locally bounded and measurable control functions, the set $\mathcal{R}_{\mathbb{1}}$ reachable from the identity is the full connected Lie subgroup $\mathbf{S}$. In fact, $\mathcal{R}_{\mathbb{1}}(\epsilon) = \mathbf{S}$ for all $\epsilon > 0$, so $\mathbf{S}$ can be reached instantaneously.*

Proof. The first statement follows from the standard fact that if $\mathcal{R}_{\mathbb{1}}$ is a subgroup then $\mathcal{R}_{\mathbb{1}} = \mathbf{S}$; to show that $\mathcal{R}_{\mathbb{1}}$ is a subgroup, showing that inverses are contained in $\mathcal{R}_{\mathbb{1}}$ follows from right-invariance. For the second statement, we use the standard trick of arbitrary rescaling that for any control function $f(t \leq T_1)$ that yields U at time T_1, the function $g(t) = \frac{T_1}{T_2} f(t\frac{T_1}{T_2})$ also yields U at time T_2; since we can take $T_2 \to 0$, we have that $\mathcal{R}_{\mathbb{1}}(T_1) = \mathcal{R}_{\mathbb{1}}(T_2)$ for all $T_1, T_2 > 0$. This completes the proof. □

Notice that this means the time-optimal control sequence consists of arbitrarily strong delta-like-pulses applied for arbitrarily short times, if there is no drift. This is in contrast with Nielsen's setting from 3.1.1 and 3.1.2, where the arbitrary rescaling of the Hamiltonian imposes the equivalence between the metric and the time T, whereas in this case we set the time $T \to 0$.

Sussmann also shows the following important fact (see for example [24] for a detailed proof):

Lemma 3.8. *Any $U \in \mathbf{S}$ can be written as a finite product of elements of the form $\exp\{-itH_k\}$, where $0 \leq k \leq m$.*

We will comment on the potential significance of this result later.

3.2.2 Khaneja, Brockett, Glaser 2000

We now reformulate key results presented in the article "Time Optimal Control in Spin Systems," published in 2000 by Khaneja, Brockett, and Glaser [25]. The key result is the introduction of the *adjoint* control system, and the intuition that the minimum time is equivalent to the minimum coset time (where we refer to the infimizing time as the minimum time to match the conventions of the literature). All control functions here lie in F_u, the space of locally bounded measurable functions defined on $[0, \infty)$ as in the setting of Jurdjevic/Sussmann.

Here, we consider a similar scenario as above, with $G = SU(2^n)$ and its associated Lie algebra $\mathfrak{g} = su(2^n)$. For $U \in G$ and some element $W \in su(2^n)$, we define the *adjoint action*

$Ad_U(W) = U^\dagger W U$, and for a set K we have $Ad_K(W) = \bigcup_{k\in K} Ad_k(W)$. For K a compact closed subgroup of G, we can write $\mathfrak{g} = \mathfrak{m} + \mathfrak{k}$.

First, it is well known that if G/K is a Riemannian symmetric space and $\mathfrak{h}$ is a Cartan subalgebra of the pair $(\mathfrak{g}, \mathfrak{k})$ with $A = \exp\{\mathfrak{h}\} \subset G$, then we can decompose $G = KAK$. This is known as the KAK-decomposition, and in some cases (such as single-qubit systems) yields a trivial bang-bang optimal sequence of a bang followed by a drift evolution, followed by a bang.

From the setting of Jurdjevic/Sussmann in [9], we can further define the infimizing (instead of minimum since we take the closure of the reachable set) time $t^*(U_F) = \inf\{t \geq 0 \mid U_F \in \overline{\mathcal{R}_{\mathbb{1}}(t)}\}$, and $t^*(KU_F) = \inf\{t \geq 0 \mid kU_F \in \overline{\mathcal{R}_{\mathbb{1}}(t)}, k \in K\}$. Similarly to Theorem (3.7), we can show that if $K = S$, the Lie subgroup corresponding to the subalgebra generated by the controls, then $t^*(U_F) = 0$ for all $U_F \in K$. The methodology used is equivalent to that as in [9], which in particular uses the following Lemma:

Lemma 3.9. *(Lemma 3 in [25]) Let $U \in G$ and $X : \mathbb{R} \to \mathfrak{g}$ be a locally bounded measurable function of time. If $X_n(t)$ converges to $X(t)$ in the sense that $\lim_{n\to\infty} \int_0^T \|X(t) - X_n(t)\|\, dt = 0$, then the solution to the differential equation $\dot{U} = X_n(t)U$ at time T converges to the solution of $\dot{U} = X(t)U$ at time T. Note that this follows as a direct consequence of the uniform convergence of the Peano-Baker series.*

This can be used to show the analogous result as in Jurdjevic/Sussmann:

Lemma 3.10. *(Lemma 4 in [25]) For any reachable unitary $U_F \in G$, we have $t^*(U_F) = t^*(KU_F)$. As such, the infimizing time to reach a particular unitary is the same as the infimizing time to reach any element in the right coset of K associated with that unitary.*

Proof. Same as in Jurdjevic/Sussmann, except with the added justification of convergence using the previous Lemma (3.9). □

From this, we can define an *adjoint control system*: let $P \in G$, and define the control system $\dot{P} = XP$, where the control X is restricted to an adjoint orbit $Ad_K(H_0) = \{k^{-1}H_0k \mid k \in K\}$. Then, we can define the *minimum coset time* $L^*(KU_F) = \inf\{t \geq 0 \mid P(t) \in KU_F\}$, which is the infimizing time required to steer the adjoint control system from the identity to the coset KU_F. This allows for the main theorem, whose full proof can be found in [25]:

Theorem 3.11. *(adapted from Theorem 7 in [25]) For any reachable unitary $U_F \in G$, we have $t^*(U_F) = L^*(KU_F)$. As such, the infimizing time to reach a particular unitary in a bilinear*

control system is the same as the infimizing time to reach the right coset of K associated with that unitary in the adjoint control system.

From the well-known result stated above, if G/K is a Riemannian symmetric space, and is in particular of rank one, then we can decompose $G = KAK$ where $A = \exp\{-i\alpha H_0\}$; this means that $t^*(U_F)$ is the smallest value $\alpha > 0$ such that $U_F = k_1 e^{-i\alpha H_0} k_2$, with $k_1, k_2 \in K$.

If instead G/K is still a Riemannian symmetric space but with rank greater than one, the time-optimal tori theorem states that $t^*(U_F)$ is the smallest value $\alpha > 0$ such that $U_F = k_1 e^{-i\alpha Y} k_2$, with $k_1, k_2 \in K$ and $Y \in \mathfrak{c}(H_0)$, where $\mathfrak{c}(H_0)$ is the convex hull of the Weyl orbit of H_0 (definitions of convex hull, Weyl orbit, and the precise theorems can be found in [25]). The time-optimal tori theorem is further discussed in "Lie Theory for Quantum Control" by Dirr and Helmke in 2008 [26]. In particular, we point out that it holds only for bilinear systems that admit a Cartan-like decomposition as above, where G/K is a Riemannian symmetric space.

3.2.3 Comments on Sussmann/Khaneja

The results by Sussmann/Khaneja effectively show, coupled with Nielsen, that the case of zero drift is "good," and is well understood. Furthermore, if G/K is a Riemannian symmetric space, then the minimum time evolution is of bang-bang form, which is also "good." For example, the case of two qubits satisfies this, and as such has been widely studied, where for example $G/K = SU(4)/\Big(SU(2) \otimes SU(2)\Big)$. Here, the single-qubit Paulis and the two-qubit Paulis each span subalgebras that yield a Cartan decomposition of $\mathfrak{g} = \mathfrak{su}(4)$. In this case, and in particular in only this case for qubit systems, the geodesic distance (the value α associated with the convex hull of the Weyl orbit) immediately corresponds to the minimum time t^*, so time-optimal evolution is easily calculable. On the other hand, if G/K is not a Riemannian symmetric space but rather sub-Riemannian (as is the case for higher-qubit systems such as $SU(2^N)/SU(2)^{\otimes N}$), the same results do not hold. Extending these to higher-qubit systems have yielded some lower bounds (see for example [27] and [28], as well as similarly inspired lower bounds as in [29, 30]), but in general a full characterization of time-optimal control for these systems remains a challenge. Studying such sub-Riemannian manifolds is a difficult task, and has been explored in particular in [31] and [32], and is included in a comprehensive review presented in [16].

On the other hand, nevertheless even in the general case the adjoint-equivalence theorem of Khaneja shows that the optimal time evolution can be considered to be alternating bang-bang sequences between the controls and drift evolution (with some caveats). However, during the

projection mapping there is no requirement to bound the number of switchings, even on whether it is finite or infinite. One might suggest by applying Lemma (3.8) regarding finite generation (see also, for example, [33]) combined with the construction of t^* as an infimizing rather than minimizing time, that assuming a finite number of switchings is a valid option. In this case, then, we can assume that there exists an infimizing-time optimal bang-bang sequence with a finite number of switchings for any $U_F \in G$. This would then transition nicely into a comparison with parameterized quantum circuits. It is also important to point out, though, that the bound on the number of such switchings for finite generation is the dimension of the Lie algebra, which scales exponentially in the number of qubits. As such, one question is whether incorporating an allowable error ϵ, such that the generation is not necessarily exact, can allow this scaling to be sub-exponential.

Another interesting point to make at this juncture is that the settings of Nielsen and Khaneja are somewhat complementary to each other with respect to their treatment of the drift Hamiltonian H_0. In Nielsen's setting, the drift is effectively excluded as the constructions require arbitrary manipulations of the total Hamiltonian. On the other hand, in Khaneja's setting it is in particular the drift that determines the minimum times, as it is effectively the one evolving in G/K. Thus, one could think of Nielsen as discussing movements within K by the controls in relation to gate sequences (for him, $K = G$), whereas Khaneja discusses movements outside of K by the drift. What is yet to be determined, then, is the relation between Khaneja's adjoint representation and minimal gate sequences discussed by Nielsen. It is the hope that with the bang-bang assumption above, we can get closer to making such connections.

3.3 Open Questions and Approaches

Here we suggest and justify the importance of a few open questions derived from these results:

1. How does the minimum time T^* scale with the allowable error ϵ? How does this scaling change when we switch between the assumptions of bang-bang optimality, piecewise-constant functions, or any locally bounded measurable functions?

2. How does the minimum number of switchings, or the minimum number of steps in the finite generation scheme, scale with the allowable error ϵ?

3. How does the minimum time T^*_M scale with the number of switchings M? That is, given bang-bang optimality and a fixed maximal number of switchings M, what is the short-

est sum of time evolutions for a sequence? How does this relationship change when an allowable error ϵ is introduced?

For each of these questions, any introduction of ϵ can either be a fixed value, or a scaling factor such as $o(1)$ or "constant in the number of qubits."

To begin answering any of these questions, one might initially try to apply Nielsen, Sussmann, and Khaneja's results to small systems of two or three qubits. As mentioned above, the case of two qubits has been well studied in the Sussmann/Khaneja context. Both Nielsen's lower- and upper- bound results are trivial in the case of two qubits, as his gate set includes all one- and two- qubit gates, so the minimum gate complexity satisfies $\mathfrak{C}_{\mathcal{G}}[U] = 1$ for all $U \in SU(4)$. Perhaps with the inclusion of PQCs, as will be justified in section 4, one could outline approaches for studying three-qubit systems under each of the models (Nielsen, Khaneja, PQC):

1. (Nielsen) Pick a three-qubit gate U for which the minimal gate sequence (consisting of one- and two- qubit gates) is known, or easily computable. Compute the geodesic distance $\mathfrak{C}_{FM}[U]$ from a Finsler metric F given by the entrywise two-norm of the instantaneous Hamiltonian, and compare this value to the simulated value.

2. (Khaneja) Pick the same three-qubit gate U, and compute its minimum finite generation (note that if the 'minimum' is determined by the count itself, then it is the same as the minimal gate sequence from Nielsen above) with respect to total time, for various three-body drift Hamiltonians H_0. Consider the case of full local control, so the evolution is on $SU(8)/SU(2)^{\otimes 3}$, and compute various minimum times depending on H_0. Then compute minimum number of switchings depending on H_0. An upper bound on these minimum times has been given, for example, by Arenz and Rabitz in 2018 [34].

3. (PQC) From the same framework as Khaneja, consider the case of full local control, and parameterize a circuit assuming a bang-bang framework, and fixing the number of switchings. Consider how the error scales with this number of switchings.

4 PQC's and Bang-Bang Time-Optimal Control

One of the most promising models for quantum computation in the modern NISQ era is that of variational quantum algorithms executed on parameterized quantum circuits, defined earlier in (2.2.2). We refer to [11] for a full review of the field, and a recent review by Magann et al. [35] that has shown relationships between parameterized quantum circuits and the quantum control

model. Furthermore, we point out that Lloyd [36] and Biamonte [37] have proven the existence of universal parameterized quantum circuits by demonstrating their ability to polynomially simulate the discrete quantum circuit model. Each of the methods utilize, at their core, two differing measures of complexity: the total number of layers in the parameterized quantum circuit, and the sum of the coefficients in the applications of each parameterized gate. These two measures correspond to the circuit depth in the discrete circuit model and the total evolution time in the controlled Schrödinger equation model respectively. In this section we propose an alternative measure of complexity for PQC's, inspired by the concept of minimizing the number of switchings in bang-bang time-optimal control, that may be better suited for comparing the various models.

Proposition 4.1. *Consider a parameterized quantum circuit $\mathcal{P}$ implementing a unitary U in the following way:*

$$U = \prod_{j=1}^{L} e^{-i\alpha_j H_{j,1}} e^{-iH_{j,2}}, \ \alpha_j \in \mathbb{R}_+ \cup \{0\}$$

Then, we propose a new measure of complexity $\mathfrak{C}_{new}[\mathcal{P}] = L \cdot \sum_{j=1}^{L} \alpha_j$ that in particular depends on both L and $\{\alpha_j\}$.

We note that $\mathfrak{C}_{new}[P]$ is effectively the *product* of the two complexities presented by Lloyd [36] and Biamonte [37]. The inclusion of the product of the terms is also inspired by the measure of complexity $T \cdot \max_{0 \leq t \leq T} \|H(t)\|$ for adiabatic quantum computation as presented in [38] (which was beyond the scope of this current work but is another commonly studied model of quantum computation).

One of the reasons behind this proposed measure of complexity as opposed to the more commonly used measure of simply counting the number of layers L is inspired by the controlled Schrödinger equation model. For example, one particular variational quantum algorithm, called the quantum approximate optimization algorithm (QAOA), is implemented on a PQC to solve combinatorial optimization problems [39]. Inspired by the aforementioned adiabatic quantum computation model, this particular PQC has $H_{j,1} = H_1$ for all j odd and $H_{j,1} = H_2$ for all j even, and $e^{-iH_{j,2}} = \mathbb{1}$ for all $j \in [1 \dots L]$ (as in the format of Proposition (4.1)), such that at each layer j one of the Hamiltonians H_1, H_2 is applied for a particular time α_j.

One can notice that this is precisely the form of bang-bang control in the controlled Schrödinger equation, with H_1 as the "drift" Hamiltonian and H_2 as the single "control" Hamiltonian. We

see that the application of $e^{-it_1 H_1}$ incurs the *same cost* as the application of $e^{-it_2 H_1}$ for all times t_1, t_2. This shines some light on showing that recent claims for $L = 1$ QAOA being able to converge and effectively solve certain computational problems [39] are somewhat misleading, since it is saying that all unitaries of the form $e^{-it_1 H_1} e^{-it_2 H_2}$ are "equally difficult/easy" to reach.

Thinking back to the time-optimal tori theorem from section 3.2.2, in the case of a Riemannian symmetric space this yields a constant measure of complexity $L = 2$ for all unitaries U, which would be a somewhat misleading concept. Nevertheless, the fact that in the case of a Riemannian symmetric space the depth of the associated PQC is constant presents a potentially interesting connection: does the depth of a PQC effectively measure, or scale with, the rank of G/K? (In the Riemannian symmetric case, the rank is 1) In this sense, even an affirmative answer to this question, which would in and of itself be of significant interest, would not yield a "proper" measure of complexity for a parameterized quantum circuit, since as a discrete measure it effectively groups together far too many PQCs with the same complexity. On the other hand, Lloyd's approach [36] of simply adding the application times $\sum \alpha_j$ can be seen to correspond to the total evolution time in the controlled Schrödinger equation model, which by itself has been seen to be difficult to connect with discrete models such as the circuit model.

As such, we are proposing here a way to combine these two measures into a "hybrid discrete/continuous" measure. One can notice that this measure of complexity is equally valid for a particular bang-bang sequence in the controlled Schrödinger equation model, since as mentioned above the form of a parameterized quantum circuit in Proposition (4.1) is precisely translatable as a bang-bang sequence. Therefore, one could also propose that the measure of complexity for a bang-bang control sequence be related to this measure as well:

Proposition 4.2. *Consider the setting of the controlled Schrödinger equation, with the bilinear control system $H(t) = H_0 + \sum_{k=1}^{m} f_k(t) H_k$, with the Lie subgroup $K = \exp\{\mathfrak{k}\}$, where $\mathfrak{k}$ is the subalgebra generated by the controls $-iH_1, \ldots, -iH_m$. Now, consider a bang-bang sequence $\mathcal{B}$ implementing a unitary U in the following way:*

$$U = \prod_{j=1}^{L} \left(e^{-it_j H_0} \cdot k_j \right) \text{ with } t_j \in \mathbb{R}_+ \cup \{0\}, \ k_j \in K$$

Then, we propose a new measure of complexity $\mathfrak{C}_{new}[\mathcal{B}] = L \cdot \sum_{j=1}^{L} t_j$.

Here, one can notice the similarity between a parameterized quantum circuit $\mathcal{P}$ and a bang-

bang sequence $\mathcal{B}$, where the only difference is that while the non-parameterized terms $e^{-iH_{j,2}}$ in $\mathcal{P}$ are pre-determined, the elements k_j in $\mathcal{B}$ can be chosen at will. Nevertheless, the components of $\mathcal{P}, \mathcal{B}$ that are included within the complexity $\mathfrak{C}_{new}$ are noticeably the same. It is the hope of this work that this intuitive measure of complexity could be an effective comparison metric for future studies of both parameterized quantum circuits and time-optimal quantum control.

5 Concluding Remarks

In this work, we have attempted to provide a method of discussing different models of quantum computation using a framework of comparing their various measures of complexity in the resources used. In particular, we have noted the difficulty of connecting discrete and continuous components such as gate count and time, as well as the importance of sub-Riemannian considerations in discussing measures of complexity that become extremely relevant due to the introduction of a drift Hamiltonian H_0 in continuous-time models of quantum computation such as the controlled Schrödinger equation. To connect these considerations with the recently growing field of variational quantum algorithms on parameterized quantum circuits, we have proposed a measure of complexity for PQCs that attempts to effectively combine both discrete and continuous metrics.

Using this new measure, as well as the connections made to other models of quantum computation, one could ask many questions worthy of further study regarding PQCs. For example, in section 3 we briefly discussed the notion of classical versus quantum resources associated with a particular quantum model. Particularly in the cases of PQCs and the controlled Schrödinger equation, the classical component of *finding* the relevant parameters (also referred to as *trainability*) is especially important. In this sense, does trainability of a circuit $\mathcal{P}$ get impacted when you fix a depth L such that the optimal solution minimizes the complexity $\mathfrak{C}_{new}[\mathcal{P}]$? In the similar realm of quantum control [35], it has been observed that such difficulties arise close to the minimum time.

Furthermore, it would be interesting to compare this new measure $\mathfrak{C}_{new}[\mathcal{P}]$ to the existing measures of "effective dimension" of a PQC [12, 13], and determine the relationships between these constructions, which are less "intuitive" and more contrived than a simple function of the number of layers or sum of coefficients. In this sense, our proposed measure of complexity can be seen as something that is "physically relevant," since it places bounds on the parameters themselves of a PQC rather than through a contrived metric (and in a way that does not just

naïvely take a sum as representing the "total time").

Before we can design new algorithms and/or models of quantum computation, it is important to understand how we are communicating results from such models, since it is hard to compare them without explicitly defining the setup and resources used. Measuring complexity through resources as we have done here is an important step in this direction, especially in defining a true "quantum advantage" as desired when comparing these models and their resources to existing complexity classes such as P and NP, or even the quantum classes BQP and QMA. As quantum algorithms in the NISQ era would likely be done from using the model of parameterized quantum circuits but implemented in a continuous-time fashion as in the controlled Schrödinger equation model, it is our hope that the points raised in this work towards a unified model of quantum computation would inspire further connections to be made, thereby allowing a true quantum advantage to be defined, justified, and achieved through execution.

References

[1] Engineering National Academies of Sciences and Medicine. *Quantum Computing: Progress and Prospects*. The National Academies Press, 2019.

[2] M. E. Cuffaro. The philosophy of quantum computing. *arXiv*, quant-ph/2103.09334, 2021.

[3] J. R. McClean, J. Romero, R. Babbush, and A. Aspuru-Guzik. The theory of variational hybrid quantum-classical algorithms. *New J. Phys.*, 18(2):023023, 2016.

[4] S. Arora and B. Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, 2009.

[5] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2010.

[6] A.Y. Kitaev, A. Shen, M.N. Vyalyi, and M.N. Vyalyi. *Classical and Quantum Computation*. American Mathematical Society, 2002.

[7] J. Watrous. *Quantum Computational Complexity*. Springer New York, 2009.

[8] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter. Elementary gates for quantum computation. *Phys. Rev. A*, 52(5):3457–3467, 1995.

[9] V. Jurdjevic and H. J. Sussmann. Control systems on Lie groups. *J. Differ. Equ.*, 12(2):313–329, 1972.

[10] J. Brylinski and R. Brylinski. Universal quantum gates. *arXiv*, quant-ph/0108062, 2001.

[11] M. Cerezo, A. Arrasmith, R. Babbush, S. C. Benjamin, S. Endo, K. Fujii, J. R. McClean, K. Mitarai, X. Yuan, L. Cincio, and P. J. Coles. Variational quantum algorithms. *arXiv*, quant-ph/2012.09265, 2020.

[12] T. Haug, K. Bharti, and M. S. Kim. Capacity and quantum geometry of parametrized quantum circuits. *arXiv*, quant-ph/2102.01659, 2021.

[13] M. Benedetti, E. Lloyd, S. Sack, and M. Fiorentini. Parameterized quantum circuits as machine learning models. *Quantum Sci. Technol.*, 4(4):043001, 2019.

[14] C. Brif, R. Chakrabarti, and H. Rabitz. Control of quantum phenomena: past, present and future. *New J. Phys.*, 12(7):075008, 2010.

[15] U. Boscain, M. Sigalotti, and D. Sugny. Introduction to the foundations of quantum optimal control. *arXiv*, quant-ph/2010.09368, 2020.

[16] S. Deffner and S. Campbell. Quantum speed limits: from Heisenberg's uncertainty principle to optimal quantum control. *J. Phys. A.*, 50(45):453001, 2017.

[17] E. Chitambar and G. Gour. Quantum resource theories. *Rev. Mod. Phys.*, 91:025001, 2019.

[18] A. Y. Kitaev. Quantum computations: algorithms and error correction. *Russ. Math. Surv.*, 52(6):1191–1249, 1997.

[19] C. M. Dawson and M. A. Nielsen. The Solovay-Kitaev algorithm. *Quantum Info. Comput.*, 6(1):81–95, 2006.

[20] M. A. Nielsen. A geometric approach to quantum circuit lower bounds. *Quantum Info. Comput.*, 6(3):213–262, 2006.

[21] M. A. Nielsen, M. R. Dowling, M. Gu, and A. C. Doherty. Quantum computation as geometry. *Science*, 311(5764):1133–1135, 2006.

[22] G.H. Golub, C.F. Van Loan, C.F. Van Loan, and P.C.F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, 1996.

[23] H. J. Sussmann. The "bang-bang" problem for certain control systems in gl(n,R). *SIAM J. Control Optim.*, 10(3):470–476, 1972.

[24] R. Zeier. *Lie–theoretischer Zugang zur Erzeugung unitärer Transformationen auf Quantenrechnern*. Universitätsverlag Karlsruhe, Karlsruhe, 2006. Zugl.: Universität Karlsruhe, Dissertation, 2006.

[25] N. Khaneja, R. Brockett, and S. J. Glaser. Time optimal control in spin systems. *Phys. Rev. A*, 63:032308, 2001.

[26] G. Dirr and U. Helmke. Lie theory for quantum control. *GAMM-Mitteilungen*, 31:59–93, 2008.

[27] R. Zeier, M. Grassl, and T. Beth. Gate simulation and lower bounds on the simulation time. *Phys. Rev. A*, 70:032319, 2004.

[28] A. M. Childs, H. L. Haselgrove, and M. A. Nielsen. Lower bounds on the complexity of simulating quantum gates. *Phys. Rev. A*, 68:052311, 2003.

[29] C. Arenz, B. Russell, D. Burgarth, and H. Rabitz. The roles of drift and control field constraints upon quantum control speed limits. *New J. Phys.*, 19(10):103015, 2017.

[30] J. Lee, C. Arenz, H. Rabitz, and B. Russell. Dependence of the quantum speed limit on system size and control complexity. *New J. Phys.*, 20(6):063002, 2018.

[31] R. Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*. American Mathematical Society, 2002.

[32] A. Agrachev, D. Barilari, and U. Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*. Cambridge University Press, 2019.

[33] D. D'Alessandro. Uniform finite generation of compact Lie groups. *Syst. Control Lett.*, 47(1):87–90, 2002.

[34] C. Arenz and H. Rabitz. Controlling qubit networks in polynomial time. *Phys. Rev. Lett.*, 120:220503, 2018.

[35] A. B. Magann, C. Arenz, M. D. Grace, T. Ho, R. L. Kosut, J. R. McClean, H. A. Rabitz, and M. Sarovar. From pulses to circuits and back again: A quantum optimal control perspective on variational quantum algorithms. *PRX Quantum*, 2:010101, 2021.

[36] Seth Lloyd. Quantum approximate optimization is computationally universal. *arXiv*, quant-ph/1812.11075, 2018.

[37] J. Biamonte. Universal variational quantum computation. *Phys. Rev. A*, 103:L030401, 2021.

[38] D. Aharonov, W. van Dam, J. Kempe, Z. Landau, S. Lloyd, and O. Regev. Adiabatic quantum computation is equivalent to standard quantum computation. In *45th Annual IEEE Symposium on Foundations of Computer Science*, 2004.

[39] E. Farhi, J. Goldstone, and S. Gutmann. A quantum approximate optimization algorithm. *arXiv*, quant-ph/1411.4028, 2014.