

# Better dual functions for Gabor time-frequency lattices

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**Abstract.** Gabor time-frequency lattices are sets of functions of the form  $g_{m\alpha, n\beta}(t) = e^{-2\pi i\alpha mt}g(t - n\beta)$  generated from a given function  $g(t)$  by discrete translations in time and frequency. It was recently observed by Wexler and Raz that the behavior of a lattice  $(m\alpha, n\beta)$  can be connected to that of a dual lattice  $(\frac{m}{\beta}, \frac{n}{\alpha})$ . We establish this interesting relationship rigorously and study its properties. We also exploit the connection between the two lattices to construct expansions having improved convergence and localization properties.

This paper reports on joint work with Henry Landau and Zeph Landau [2]. It concerns expansions of  $L^2(\mathbb{R})$ -functions  $f(x)$  into families  $g_{m\alpha, n\beta}(x)$  obtained by translating and modulating a fixed function  $g$  in  $L^2(\mathbb{R})$ ,  $g_{m\alpha, n\beta}(x) = e^{-2\pi im\alpha x}g(x - n\beta)$ , with  $\alpha, \beta > 0$  fixed, and  $m, n$  ranging over  $\mathbf{Z}$ . We call such families *Gabor time-frequency lattices* (or *Gabor lattices* for short), after an expansion of this type proposed by Gabor in [4] (where  $g$  was a Gaussian, and  $\alpha\beta = 1$ ). The general problem of identifying coefficients  $c_{m,n}(f)$  so that

$$f(x) = \sum_{m,n} c_{m,n} g_{m\alpha, n\beta}(x) \quad (1)$$

has been discussed in many places. In [1] this problem was studied in the context of *frames*; the  $g_{m\alpha, n\beta}$  are said to constitute a frame if there exist  $A > 0$  and  $B < \infty$  such that, for all  $f \in L^2(\mathbb{R})$ ,  $A\|f\|^2 \leq \sum_{m,n} |\langle f, g_{m\alpha, n\beta} \rangle|^2 \leq B\|f\|^2$ , where  $\langle \cdot, \cdot \rangle$  denotes the standard  $L^2$ -inner product,  $\langle f, g \rangle = \int f(x)\overline{g(x)} dx$ , and  $\|f\| = \langle f, f \rangle^{\frac{1}{2}}$ . If the  $g_{m\alpha, n\beta}$  constitute a frame, then one can show (see [3]) that the least squares choice for the  $c_{m,n}$  in the expansion for  $f$  is given by

$$c_{m,n} = \langle f, S_{g; \alpha, \beta}^{-1} g_{m\alpha, n\beta} \rangle = \langle f, \tilde{g}_{m\alpha, n\beta} \rangle, \quad (2)$$

where  $S_{g;\alpha,\beta}$  is the *frame operator*  $S_{g;\alpha,\beta} f = \sum_{m,n} \langle f, g_{m\alpha,n\beta} \rangle g_{m\alpha,n\beta}$ , and where the  $\tilde{g}_{m\alpha,n\beta}$  are derived from the single function  $\tilde{g} = S_{g;\alpha,\beta}^{-1} g$  by the same time-frequency translations as the  $g_{m\alpha,n\beta}$  from  $g$ . We refer to  $\tilde{g}$  as the *frame dual function*.

A different approach to finding appropriate  $c_{m,n}$  was taken in [8]. (Strictly speaking, [8] was concerned with discrete sequences in  $\ell^2$  rather than  $L^2$ -functions; see also [6]). It turns out that, at least for “nice”  $f, g, h \in L^2$ ,

$$\sum_{m,n} \langle h, g_{m\alpha,n\beta} \rangle f_{m\alpha,n\beta} = \frac{1}{\alpha\beta} \sum_{m,n} \langle f, g_{\frac{m}{\beta}, \frac{n}{\alpha}} \rangle h_{\frac{m}{\beta}, \frac{n}{\alpha}} .$$

We call this remarkable identity the *Wexler-Raz identity*. If we introduce the notation  $T_{g;\alpha,\beta}$  for the operator from  $L^2(\mathbb{R})$  to  $\ell^2(\mathbf{Z}^2)$  that maps  $f$  to  $(\langle f, g_{m\alpha,n\beta} \rangle)_{m,n \in \mathbf{Z}}$ , then the Wexler-Raz identity can also be rewritten as

$$T_{h;\alpha,\beta}^* T_{g;\alpha,\beta} f = \frac{1}{\alpha\beta} T_{f;\frac{1}{\beta}, \frac{1}{\alpha}}^* T_{g;\frac{1}{\beta}, \frac{1}{\alpha}} h . \quad (3)$$

This identity holds whenever all the operators involved are bounded and even in some cases where some of the T-operators are unbounded (see [2]). If  $h$  satisfies  $T_{g;\frac{1}{\beta}, \frac{1}{\alpha}} h = \alpha\beta \ell_{0,0}$ , where  $\ell_{k,\ell}$  is the sequence with entries  $(\ell_{k,\ell})_{m,n} = \delta_{k,m} \delta_{\ell,n}$ , or equivalently, if

$$\langle h, g_{\frac{m}{\beta}, \frac{n}{\alpha}} \rangle = \alpha\beta \delta_{m,0} \delta_{n,0} , \quad (4)$$

then the coefficients  $c_{m,n} = \langle f, h_{m\alpha,n\beta} \rangle$  will satisfy (1). Among all these different “biorthogonal” choices  $h$ , one can choose the one with minimal  $L^2$ -norm; we call it the *Wexler-Raz dual function* and denote it by  $g^\#$ .

As the functions  $\tilde{g}$  and  $g^\#$  minimize different things, it is not immediately clear how they are related. It turns out that they are identical.

The following argument shows why. (Note that this is only a sketch. For rigorous and detailed arguments, see the three very different treatments in [2], [5], and [7].) First of all, note that by the general theory of frames,  $S_{\tilde{g};\alpha,\beta} = (S_{g;\alpha,\beta})^{-1}$ , so that  $\tilde{g} = S_{\tilde{g};\alpha,\beta} g = T_{\tilde{g};\alpha,\beta}^* T_{g;\alpha,\beta} g$ . By the Wexler-Raz identity, this implies  $\tilde{g} = \frac{1}{\alpha\beta} T_{g;\frac{1}{\beta}, \frac{1}{\alpha}}^* T_{\tilde{g};\frac{1}{\beta}, \frac{1}{\alpha}} \tilde{g}$ , from which we retain that  $\tilde{g} \in \text{Ran} T_{g;\frac{1}{\beta}, \frac{1}{\alpha}}^* = (\text{Ker} T_{g;\frac{1}{\beta}, \frac{1}{\alpha}})^\perp$ . On the other hand,  $T_{\tilde{g};\alpha,\beta}^* T_{g;\alpha,\beta} = \text{Id}$ , which implies, by the Wexler-Raz identity,  $T_{g;\frac{1}{\beta}, \frac{1}{\alpha}} \tilde{g} = \frac{1}{\alpha\beta} \ell_{0,0}$ . Because  $\tilde{g} \perp \text{Ker} T_{g;\frac{1}{\beta}, \frac{1}{\alpha}}$ , it follows that  $\tilde{g}$  is the unique minimal  $L^2$ -solution to this equation, or  $\tilde{g} \equiv g^\#$ .

Finally, part of the motivation of Wexler and Raz in [8] was that, in practice, the  $g^\#$  of minimal  $L^2$ -norm may *not* be the preferred choice.

They present several ad hoc constructions for examples where, for instance, a different “biorthogonal” function leads to better time concentration. The following is a systematic approach to find biorthogonal functions that optimize other than  $L^2$ -norms.

Assume that we are interested in finding the dual function  $g^b$  for which  $\|g^b\| = \|\Lambda g^b\|$  is minimal, for example,  $\Lambda = \left(-\frac{d^2}{dx^2} + x^2\right)^{\frac{1}{2}}$  or  $\Lambda = \left(-\frac{d^2}{dx^2} + 1\right)^{\frac{1}{2}}$ . This is equivalent to finding the function  $G$  with smallest  $L^2$ -norm that satisfies  $T_{g;\frac{1}{\beta},\frac{1}{\alpha}}\Lambda^{-1}G = \alpha\beta e_{0,0}$ , and then taking  $g^b = \Lambda^{-1}G$ . Note that we are implicitly assuming that  $\Lambda$  has a bounded inverse  $\Lambda^{-1}$ . This is not a severe restriction: we can assume  $\Lambda \geq 0$  without loss of generality (otherwise, replace  $\Lambda$  by  $(\Lambda^*\Lambda)^{\frac{1}{2}}$ ), and we can then add 1 to  $\Lambda$ , if necessary, without changing the nature of the smoothness or decay constraint. We shall systematically assume  $\Lambda \geq \text{Id}$  in what follows. The minimal  $L^2$ -solution  $G$  to this equation is given by  $\alpha\beta \Lambda^{-1} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^* (T_{g;\frac{1}{\beta},\frac{1}{\alpha}} \Lambda^{-2} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^*)^{-1} e_{0,0}$ . It is however not clear in what sense this should be understood, since the operator  $T_{g;\frac{1}{\beta},\frac{1}{\alpha}} \Lambda^{-2} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^*$ , although well-defined and bounded on  $\ell^2(\mathbf{Z}^2)$ , is generally not invertible, even if  $T_{g;\frac{1}{\beta},\frac{1}{\alpha}} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^*$  is. The solution is to introduce an extra operator  $\Omega$  on  $\ell^2(\mathbf{Z}^2)$ , typically also unbounded, which complements on  $\ell^2(\mathbf{Z}^2)$  the action of  $\Lambda$  on  $L^2(\mathbb{R})$ . In other words, we replace the equation by

$$\Omega T_{g;\frac{1}{\beta},\frac{1}{\alpha}} \Lambda^{-1} G = \alpha\beta \Omega e_{0,0} ,$$

where  $\Omega$  is chosen so that  $\Omega T_{g;\frac{1}{\beta},\frac{1}{\alpha}} \Lambda^{-2} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^* \Omega$  is invertible with bounded inverse. The minimal  $L^2$ -solution  $G$  to this equation is given by  $\alpha\beta \Lambda^{-1} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^* \Omega (\Omega T_{g;\frac{1}{\beta},\frac{1}{\alpha}} \Lambda^{-2} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^* \Omega)^{-1} \Omega e_{0,0}$ . Applying  $\Lambda^{-1}$  once more then gives the desired  $\tilde{g}_\Lambda$  which is a dual function for the frame  $g_{m\alpha,n\beta}$ , and which minimizes  $\|\Lambda g^b\|$ ,

$$\tilde{g}_\Lambda = \alpha\beta \Lambda^{-2} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^* \Omega \left( \Omega T_{g;\frac{1}{\beta},\frac{1}{\alpha}} \Lambda^{-2} T_{g;\frac{1}{\beta},\frac{1}{\alpha}}^* \Omega \right)^{-1} \Omega e_{0,0} .$$

An explicit example where this approach leads to better frequency concentration can be found in [2].

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