Regularity of Refinable Function Vectors
First draft

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Subject classification:

Keywords: Shift–invariant spaces.

1. Introduction

In this paper we shall discuss the smoothness of refinable function vectors. These are solutions to functional equations of the type

$$\boldsymbol{\phi}(x) = \sum_{n=0}^{N} \boldsymbol{P}_n \, \boldsymbol{\phi}(2x-n) \ , \tag{1.1}$$

where the "coefficients" $\boldsymbol{P}_n$ are $r \times r$ matrices ($r \in \mathbb{N}$, $r \geq 1$), and where $\boldsymbol{\phi} := (\phi_0, \ldots, \phi_{r-1})^T$ is an r–dimensional function vector. Equations of type (1.1) are natural generalizations of the refinement equations studied in e.g. Cavaretta, Dahmen, and Micchelli (1991), where $r = 1$; therefore we shall call them *refinement equations* as well, or occasionally *vector refinement equations*.

Vector refinement equations have come up in several papers. The oldest example is probably the multiwavelet construction by Alpert and Rokhlin (1991) (see also Alpert (1993)), where the ϕ_ν are all supported on $[0, 1]$ and are polynomials of degree $r - 1$ on their support. In this example the smoothness of the ϕ_ν is of course known; equation (1.1) is useful as a computational tool in going from one multiresolution level to the next. Matrix generalizations of type (1.1) were also discussed in more generality in Goodman, Lee, and Tang (1993) and Goodman and Lee (1994), including how to define wavelets once the scaling functions were known. However, it was not clear how to construct smooth non-polynomial examples, let alone how to connect smoothness with properties of the $\boldsymbol{P}_n$. This was in marked contrast with the case $r = 1$, where the link between smoothness of $\boldsymbol{\phi}$ and properties of the $\boldsymbol{P}_n$ or of the *refinement mask*

$$\boldsymbol{P}(u) := \frac{1}{2} \sum_n \boldsymbol{P}_n \, e^{-iun} \tag{1.2}$$

is well understood, and where this connection can be exploited to construct $\boldsymbol{\phi}$ with arbitrary pre–assigned smoothness as well as many other properties (see Daubechies (1992)). Donovan, Geronimo, Hardin, and Massopust (1994) (hereafter referred to as DGHM) were the first to construct continuous non-polynomial refinable function vectors. They gave examples of special bases of selfsimilar wavelets, generated by continuous scaling functions that satisfy an equation of type (1.1). In their paper, the iterated function technique used in the construction was the key to derive smoothness, rather than properties of the $\boldsymbol{P}_n$. This first example triggered several other constructions (e.g. Strang and Strela (1994)), as well as work on the filter bank implications of (1.1) (Vetterli and Strang (1994), Heller et al. (1994)) and a systematic study of the approximation order of solutions of (1.1) (Heil, Strang, and Strela (1994), Plonka (1995)). This last work contains the key to understanding how solutions of (1.1) can be smooth.

As shown in Plonka (1995), the space spanned by the functions $\phi_\nu(x-n)$ $(n \in \mathbb{N})$ can only have approximation order m if $\boldsymbol{P}(u)$ has certain particular factorization properties. (We assume that the $\phi_\nu(x-n)$ are also L^2–stable.) This is reminiscent of the case $r=1$, where similarly L^2–stable translates of a refinable function ϕ can only provide approximation order m if the refinement mask, often denoted by $m_0(u)$, can be factored as

$$m_0(u) = \left(\frac{1+e^{-iu}}{2}\right)^m q(u) \ , \tag{1.3}$$

where $q(0) = 1$, q is 2π–periodic and non–singular for $u = \pi$. By iterating the formula

$$\hat{\phi}(u) = m_0\left(\frac{u}{2}\right) \hat{\phi}\left(\frac{u}{2}\right) \ ,$$

which is obtained by Fourier transformation from (1.1), and exploiting the factorization (1.3), one then finds

$$|\hat{\phi}(u)| = |\prod_{j=1}^{\infty} m_0(2^{-j}\,u)| \le C\,(1+|u|)^{-m} \prod_{j=1}^{\infty} |q(2^{-j}\,u)| \ ,$$

where the infinite products converge uniformly on compact sets if m_0 or, equivalently, q is Hölder continuous in $u=0$. Together with estimates of the type $\sup_u|q(u)| \le B$ or, more generally, $\sup_u|q(2^{k-1}u)\,q(2^{k-2}u)\ldots q(u)| \le B^k$ for some $k \in \mathbb{N}\setminus\{0\}$, this leads to

$$|\hat{\phi}(u)| \le C\,(1+|u|)^{-m+\log_2 B} \tag{1.4}$$

(see e.g. Daubechies (1988, 1992)). The factorization (1.3), together with estimates on the quotient $q(u)$, therefore leads to decay for $\hat{\phi}$, and hence to smoothness estimates for ϕ. By using more sophisticated methods involving transfer operators, one can refine the brute force estimates (1.4) and formulate necessary and sufficient conditions on $q(u)$ ensuring that ϕ lies in some Sobolev space W^s (see Conze and Raugi (1990), Cohen and Conze (1992), Villemoes (1994), Eirola (1992), Gripenberg

(1993), Hervé (1994), Cohen and Daubechies (1994)). Here again, the factorization (1.3) is a key ingredient.

In this paper, we shall see that the factorization for the matrix $\boldsymbol{P}(u)$ discovered by Plonka (1995) for the case $r > 1$, can play a similar role, although the discussion is more intricate.
We shall assume that the $\phi_\nu(x-n)$, $\nu = 0, \ldots, r-1$, $n \in \mathbb{Z}$ form a Riesz basis for their closed linear span V_0, and that they provide approximation order m, i.e., for $f \in W^m$ one has

$$\|f - \mathrm{Proj}_{V_j} f\|_{L^2} \leq C\, 2^{-jm}\, \|f\|_{W^m},$$

where V_j is the scaled space

$$V_j = \{g \in L^2(\mathbb{R});\ g(2^{-j}\cdot) \in V_0\}.$$

Then, it is shown in Plonka (1995) that there exist $r \times r$ matrices $\boldsymbol{C}_0(u), \ldots, \boldsymbol{C}_{m-1}(u)$ (constructed explicitly in Plonka (1995); see also below) such that $\boldsymbol{P}(u)$, defined in (1.2), factors as

$$\boldsymbol{P}(u) = \frac{1}{2^m}\, \boldsymbol{C}_0(2u) \ldots \boldsymbol{C}_{m-1}(2u)\, \boldsymbol{P}^{(m)}(u)\, \boldsymbol{C}_{m-1}(u)^{-1} \ldots \boldsymbol{C}_0(u)^{-1}, \tag{1.5}$$

where $\boldsymbol{P}^{(m)}(u)$ is well–defined. By Fourier transform of (1.1) we obtain

$$\hat{\boldsymbol{\phi}}(u) = \boldsymbol{P}\left(\frac{u}{2}\right)\, \hat{\boldsymbol{\phi}}\left(\frac{u}{2}\right). \tag{1.6}$$

This can be iterated again, and we find

$$\hat{\boldsymbol{\phi}}(u) = \boldsymbol{P}\left(\frac{u}{2}\right)\, \boldsymbol{P}\left(\frac{u}{4}\right) \ldots \boldsymbol{P}\left(\frac{u}{2^n}\right)\, \hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right)\ . \tag{1.7}$$

Substituting (1.5) in (1.7) leads to

$$\begin{aligned} \hat{\boldsymbol{\phi}}(u) \;&=\; 2^{-mn}\, \boldsymbol{C}_0(u) \ldots \boldsymbol{C}_{m-1}(u) \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \\ &\quad \times \boldsymbol{C}_{m-1}\left(\frac{u}{2^n}\right)^{-1} \ldots \boldsymbol{C}_0\left(\frac{u}{2^n}\right)^{-1}\, \hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right)\ . \end{aligned} \tag{1.8}$$

Even at this stage, the case $r > 1$ is more complicated than $r = 1$. The matrices $\boldsymbol{P}(2^{-j}u)$ or $\boldsymbol{P}^{(m)}(2^{-j}u)$ do not commute, and the discussion of the convergence of an infinite product definition for $\hat{\boldsymbol{\phi}}(u)$ is therefore more complex.
Hervé (1994) studied the convergence of the matrices $\boldsymbol{\Pi}_n(u)$

$$\boldsymbol{\Pi}_n(u) := \boldsymbol{P}\left(\frac{u}{2}\right)\, \boldsymbol{P}\left(\frac{u}{4}\right) \ldots \boldsymbol{P}\left(\frac{u}{2^n}\right)\ , \tag{1.9}$$

as $n \to \infty$, and showed that convergence is assured if e.g. $\boldsymbol{P}(0) = \mathrm{diag}(1, \mu_1, \ldots, \mu_{r-1})$, with $|\mu_l| < 1$ for $l = 1, \ldots, r-1$, or if $\boldsymbol{P}(0)$ is similar to such a matrix, i.e.,

$\boldsymbol{P}(0) = \boldsymbol{M}\,\mathrm{diag}(1, \mu_1, \ldots, \mu_{r-1})\,\boldsymbol{M}^{-1}$ for some non–singular $\boldsymbol{M}$. This already excludes the case where $\boldsymbol{P}(0)$ is not diagonalizable. Moreover, our matrices $\boldsymbol{P}^{(m)}(0)$ may well have a spectral radius larger than 1, so that Hervé's results cannot be used for the products $\boldsymbol{P}^{(m)}(u/2)\ldots\boldsymbol{P}^{(m)}(2^{-n}u)$ in (1.7). Heil and Colella (1994) discuss not only the convergence of $\boldsymbol{\Pi}_n(u)$ (with results similar to Hervé (1994)) but also the convergence of $\boldsymbol{\Pi}_n(u)\boldsymbol{v}$, where $\boldsymbol{v}$ is a fixed r–dimensional vector. If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{P}(0)$ with eigenvalue 1, then $\boldsymbol{\Pi}_n(u)\,\boldsymbol{v}$ may converge even if the spectral radius ρ_0 of $\boldsymbol{P}(0)$ is > 1; Heil and Colella call this *constrained convergence*. They prove constrained convergence if $\rho_0 < 2$ and if the largest eigenvalue of $\boldsymbol{P}(0)$ is nondegenerate. We use a different technique that proves convergence of $\boldsymbol{\Pi}_n(u)\,\boldsymbol{v}$ if $\rho_0 < 2$, without non–degeneracy condition, and that extends to some cases where $\rho_0 \geq 2$, if $\boldsymbol{P}(u)$ is Hölder continuous in $u = 0$ with Hölder exponent > 1. Once convergence of (1.7) or (1.8) is established, we can proceed to the main topic of this paper, namely how the factorization (1.5), together with estimates on $\boldsymbol{P}^{(m)}(\mathrm{u})$ can lead to decay of $|\hat{\phi}_\nu(u)|$ ($\nu = 0, \ldots, r-1$) as $|u| \to \infty$. As in the case $r = 1$, this can be exploited to prove L^2–convergence and pointwise convergence theorems (in the "x–domain") similar to those in Daubechies (1988).

This paper is organized as follows. In section 2, we recall the precise results on the factorization of $\boldsymbol{P}(u)$ obtained in Plonka (1995). We also show that this factorization is necessary in order to obtain smooth functions $\phi_0, \ldots, \phi_{r-1}$. In section 3, we discuss the pointwise convergence of $\boldsymbol{\Pi}_n(u)\,\boldsymbol{a}$, as $n \to \infty$, for a fixed vector $\boldsymbol{a}$. In section 4, we exploit the factorization (1.5) to prove, under certain additional conditions, that $\lim_{n\to\infty} \boldsymbol{\Pi}_n(u)\,\boldsymbol{a}$ decays, as a function of u, for $|u| \to \infty$. Section 5 gives a short uniqueness discussion: the previous sections have constructed an infinite product solution for (1.6); if this has sufficient decay, then its inverse Fourier transform gives a solution to (1.1). Theorem 5.1 shows that, under certain conditions on the mask, this solution is unique in a wide class of functions. In section 6 we show how the decay proved in section 4 can be used to translate the pointwise convergence of $\boldsymbol{\Pi}_n(u)\boldsymbol{a}$ to L^2– and L^1–convergence, which give then L^2– and pointwise convergence in the "x–domain." Finally, section 7 studies several examples; we apply our analysis to see how the (known) smoothness of spline examples and of the DGHM scaling functions can be recovered, and we construct some new examples with controlled smoothness.

2. Factorization of the refinement mask

We want to recall some results of Plonka (1995). We start by some definitions. Let $r \in \mathbb{N}$ be fixed, and let $\boldsymbol{y} \in \mathbb{R}^r$ be a vector of length r with $\boldsymbol{y} \neq \mathbf{0}$. Here and in the following, $\mathbf{0}$ denotes the zero vector of length r. We suppose that $\boldsymbol{y}$ is of the form

$$\boldsymbol{y} = (y_0, \ldots, y_{l-1}, 0, \ldots, 0)^T \tag{2.1}$$

with $1 \leq l \leq r$ and $y_\nu \neq 0$ for $\nu = 0, \ldots, l-1$. Introducing the direct sum of quadratic matrices $\boldsymbol{A} \oplus \boldsymbol{B} := \mathrm{diag}(\boldsymbol{A},\, \boldsymbol{B})$, we define the matrix $\boldsymbol{C_y}$ by

$$\boldsymbol{C_y}(u) := \tilde{\boldsymbol{C}}_{\boldsymbol{y}}(u) \oplus \boldsymbol{I}_{r-l}, \tag{2.2}$$

where $\boldsymbol{I}_{r-l}$ is the $(r-l) \times (r-l)$ unit matrix. If $l > 1$, then $\tilde{\boldsymbol{C}}_{\boldsymbol{y}}(u)$ is defined by

$$\tilde{\boldsymbol{C}}_{\boldsymbol{y}}(u) := \begin{pmatrix} y_0^{-1} & -y_0^{-1} & 0 & \dots & 0 \\ 0 & y_1^{-1} & -y_1^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & y_{l-2}^{-1} & -y_{l-2}^{-1} \\ -e^{-iu}/y_{l-1} & 0 & \dots & 0 & y_{l-1}^{-1} \end{pmatrix} \quad ; \tag{2.3}$$

if $l = 1$, i.e. if $\boldsymbol{y} := (y_0, 0, \dots, 0)^T$ with $y_0 \neq 0$, then $\tilde{\boldsymbol{C}}_{\boldsymbol{y}}(u)$ is the scalar $(1-e^{-iu})/y_0$, so that $\boldsymbol{C_y}(u)$ is a diagonal matrix of the form

$$\boldsymbol{C_y}(u) := \mathrm{diag}\ \left(\frac{1-e^{-iu}}{y_0}, 1, \dots, 1\right) \ .$$

It can easily be observed that $\boldsymbol{C_y}(u)$ is invertible for $u \neq 0$. Further, the matrix $\boldsymbol{C_y}$ is chosen such that

$$\boldsymbol{y}^T \boldsymbol{C_y}(0) = \boldsymbol{0}^T.$$

We introduce

$$\boldsymbol{E_y}(u) := (1-e^{-iu})\, \boldsymbol{C}_{\boldsymbol{y}}^{-1}(u) \ . \tag{2.4}$$

Assuming that $\boldsymbol{y}$ is of the form (2.1), we obtain that $\boldsymbol{E_y}(u) = \tilde{\boldsymbol{E}}_{\boldsymbol{y}}(u) \oplus (1-e^{-iu})\, \boldsymbol{I}_{r-l}$ with

$$\tilde{\boldsymbol{E}}_{\boldsymbol{y}}(u) := \begin{pmatrix} y_0 & y_1 & y_2 & \dots & y_{l-1} \\ y_0 z & y_1 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & y_{l-1} \\ y_0 z & y_1 z & \ddots & y_{l-2} & y_{l-1} \\ y_0 z & y_1 z & \dots & y_{l-2} z & y_{l-1} \end{pmatrix} \qquad (z := e^{-iu}) \ . \tag{2.5}$$

Note that $\boldsymbol{E_y}$ can be written in the form

$$\boldsymbol{E_y}(u) = \boldsymbol{E_y}(0) - i(1-e^{-iu})\, (\mathrm{D}\boldsymbol{E_y})(0) \ , \tag{2.6}$$

where D denotes the differential operator with respect to ω, $\mathrm{D} := \ \mathrm{d}/\ \mathrm{d}\omega$. The vector $\boldsymbol{y}$ need not to be such that its zero entries always occur at the tail. If the non–zero entries of the vector $\boldsymbol{y}$ are given in a different order than in (2.1), then the matrices $\boldsymbol{C_y}$ and $\boldsymbol{E_y}$ are defined just by reshuffling the rows and columns accordingly.
We can now formulate the factorization results for $\boldsymbol{P}(u)$. The following theorem is proved in Plonka (1995):

Theorem 2.1 *Let* $\boldsymbol{\phi} := (\boldsymbol{\phi}_\nu)_{\nu=0}^{r-1}$ *be a refinable vector of compactly supported functions, and let* $\{\phi_\nu(\cdot - n) : n \in \mathbb{Z}, \nu = 0, \ldots, r-1\}$ *form a Riesz basis of their closed linear span* V_0. *Then* V_0 *provides approximation order* m *if and only if the refinement mask* $\boldsymbol{P}$ *of* $\boldsymbol{\phi}$ *satisfies the following conditions:*
The elements of $\boldsymbol{P}$ *are trigonometric polynomials, and there are vectors* $\boldsymbol{y}_k \in \mathbb{R}^r$; $\boldsymbol{y}_0 \neq \mathbf{0}$ $(k = 0, \ldots, m-1)$ *such that for* $n = 0, \ldots, m-1$ *we have*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^{\mathrm{T}} \, (2i)^{k-n} \, (\mathrm{D}^{n-k} \boldsymbol{P})(0) &= 2^{-n} \, (\boldsymbol{y}_n)^{\mathrm{T}}, \\ \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^{\mathrm{T}} \, (2i)^{k-n} \, (\mathrm{D}^{n-k} \boldsymbol{P})(\pi) &= \mathbf{0}^{\mathrm{T}} \, . \end{aligned} \tag{2.7}$$

Furthermore, the equalities (2.7) *imply that there are vectors* $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{r-1}$ $(\boldsymbol{x}_k \neq \mathbf{0}, \, k = 0, \ldots, m-1)$ *such that* $\boldsymbol{P}$ *factorizes*

$$\boldsymbol{P}(u) = \frac{1}{2^m} \boldsymbol{C}_{\boldsymbol{x}_0}(2u) \, \ldots \, \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u) \, \boldsymbol{P}^{(m)}(u) \, \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)^{-1} \, \ldots \, \boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1} \, , \tag{2.8}$$

where the $(r \times r)$*–matrices* $\boldsymbol{C}_{\boldsymbol{x}_k}$ *are defined by* $\boldsymbol{x}_k$ $(k = 0, \ldots, m-1)$ *via* (2.2) *and* $\boldsymbol{P}^{(m)}(u)$ *is an* $(r \times r)$*–matrix with trigonometric polynomials as entries.*

The vectors $\boldsymbol{x}_l (l = 0, \ldots, m-1)$ in Theorem 2.1 are completely defined in terms of the vectors $\boldsymbol{y}_k$ $(k = 0, \ldots, m-1)$. In particular, we have $(\boldsymbol{x}_0)^T = (\boldsymbol{y}_0)^T$, $(\boldsymbol{x}_1)^T = (-i)(\boldsymbol{y}_0)^T \, (\mathrm{D}\boldsymbol{C}_{\boldsymbol{y}_0})(0) + (\boldsymbol{y}_1)^T \boldsymbol{C}_{\boldsymbol{y}_0}(0)$ (cf. Plonka (1995)). With the assumptions in Theorem 2.1, approximation order m is equivalent with exact reproduction of algebraic polynomials of degree $< m$ in V_0. Vice versa, if algebraic polynomials of degree $< m$ can be exactly reproduced in V_0, i.e., if there are vectors $\boldsymbol{y}_l^n \in \mathbb{R}^r$ $(l \in \mathbb{Z}, n = 0, \ldots, m-1)$ such that

$$\sum_{l \in \mathbb{Z}} (\boldsymbol{y}_l^n)^T \, \boldsymbol{\phi}(x - l) = x^n \qquad (x \in \mathbb{R}; \, n = 0, \ldots, m-1) \, ,$$

then $\boldsymbol{y}_l^n$ can be written in the form

$$\boldsymbol{y}_l^n = \sum_{k=0}^{n} \binom{n}{k} \, l^{n-k} \, \boldsymbol{y}_0^k \, ,$$

and the vectors $\boldsymbol{y}_0^k$ $(k = 0, \ldots, m-1)$ satisfy the equalities (2.7) with respect to the refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$.

Now, assume that $\boldsymbol{\phi}$ is a refinable function vector with a refinement mask $\boldsymbol{P}$ satisfying the conditions (2.7) for the vectors $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{m-1}$ $(\boldsymbol{y}_0 \neq 0)$. Further, let $\boldsymbol{M} \in \mathbb{R}^{r \times r}$ be an invertible matrix and

$$\boldsymbol{\phi}^\sharp(x) := \boldsymbol{M} \, \boldsymbol{\phi}(x).$$

Then $\boldsymbol{\phi}^\sharp$ is also a refinable function vector with the refinement mask $\boldsymbol{P}^\sharp(u) := \boldsymbol{M}\,\boldsymbol{P}(u)\,\boldsymbol{M}^{-1}$, since

$$\begin{aligned}\hat{\boldsymbol{\phi}}^\sharp(u) &= \boldsymbol{M}\,\hat{\boldsymbol{\phi}}(u) = \boldsymbol{M}\,\boldsymbol{P}(u/2)\,\hat{\boldsymbol{\phi}}(u/2)\\ &= \boldsymbol{M}\,\boldsymbol{P}(u/2)\,\boldsymbol{M}^{-1}\,\hat{\boldsymbol{\phi}}^\sharp(u/2)\ .\end{aligned}$$

Observe that $\boldsymbol{P}^\sharp$ is obtained by a similarity transformation from $\boldsymbol{P}$, i.e., $\boldsymbol{P}$ and $\boldsymbol{P}^\sharp$ possess the same spectrum. Furthermore, $\boldsymbol{P}^\sharp(u)$ satisfies the conditions (2.7) for $n = 0, \ldots, m-1$ with vectors $\boldsymbol{y}_0^\sharp, \ldots, \boldsymbol{y}_{m-1}^\sharp$, given by

$$(\boldsymbol{y}_\nu^\sharp)^T = (\boldsymbol{y}_\nu)^T\,\boldsymbol{M}^{-1} \quad (\nu = 0, \ldots, m-1)\ .$$

Hence, $\boldsymbol{P}^\sharp$ can also be factored as in (2.8) with $\boldsymbol{C}$–matrices defined by certain vectors $\boldsymbol{x}_0^\sharp, \ldots, \boldsymbol{x}_{m-1}^\sharp$. In particular, we have $(\boldsymbol{x}_0^\sharp)^T := (\boldsymbol{y}_0^\sharp)^T = (\boldsymbol{y}_0)^T\,\boldsymbol{M}^{-1}$. Note that this implies that the factorization (2.8) is not invariant under basis transformations. For instance, in the case where we consider a single factorization,

$$\boldsymbol{P}(u) = \frac{1}{2} C_{y_0}(2u) P^{(1)}(u) C_{y_0}(u)^{-1}\ , \tag{2.9}$$

we could choose instead to carry out first the basis transformation

$$M = \begin{pmatrix} y_0 & y_1 & y_2 & \ldots & y_{r-1} \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}\ . \tag{2.10}$$

For $\boldsymbol{P}^\sharp(u) = \boldsymbol{M}\boldsymbol{P}(u)\boldsymbol{M}^{-1}$ the equation (2.7) now holds with $(y_0^\sharp)^T = (1, 0, \ldots, 0)$, and we can factor $\boldsymbol{P}^\sharp(u)$ accordingly. Multiplying the factored expression by $\boldsymbol{M}^{-1}$ on the left and $\boldsymbol{M}$ on the right, we obtain

$$\boldsymbol{P}(u) = \frac{1}{2} \boldsymbol{D}_{y_0}(2u) \boldsymbol{Q}^{(1)}(u) \boldsymbol{D}_{y_0}(u)^{-1}\ , \tag{2.11}$$

where $\boldsymbol{D}_y(u)$ is now defined by

$$\boldsymbol{D}_y(u) = \begin{pmatrix} 1 - z & -\frac{y_1}{y_0} z & -\frac{y_2}{y_0} z & \ldots & -\frac{y_{\pi-1}}{y_0} z \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}\ . \tag{2.12}$$

Other choices of $\boldsymbol{M}$ would lead to yet other factorizations. In most applications, the original factorization (2.8) turns out to be the most useful. We shall use the existence of this different factorization (2.11) (2.12) as a tool to study the spectrum of $\boldsymbol{P}^{(1)}(0)$.

3. Convergence of infinite matrix products

Ultimately, we are interested in L^1-solutions $\boldsymbol{\phi}(x)$ of (1.1), and their smoothness, if they have any. We also want the space spanned by the $\phi_\nu(x-n), \nu = 0. \ldots, r-1, n \in \mathbb{Z}$ to have a certain approximation order. For $\boldsymbol{\phi} \in L^1$, the Fourier transform $\hat{\boldsymbol{\phi}}$ is a well-defined and continuous vector-valued function that must satisfy (1.6) for all u. In particular, we must have

$$\hat{\boldsymbol{\phi}}(0) = \boldsymbol{P}(0)\hat{\boldsymbol{\phi}}(0) \ .$$

On the other hand, if we want any non-zero approximation order, then we must have $\hat{\boldsymbol{\phi}}(0) \neq 0$, since $\hat{\boldsymbol{\phi}}(0) \neq 0$ would imply $\int \phi_\nu(x-n)dx = 0$ for all ν, n, making it impossible to construct the function 1 as a combination of the $\phi_\nu(x-n)$. Together, these two observations imply that we should take $\hat{\boldsymbol{\phi}}(0) = \boldsymbol{a}$, where $\boldsymbol{a}$ is a left eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. Note that we know that 1 has to be an eigenvalue of $\boldsymbol{P}(0)$ because of (2.7). In all the examples we shall consider in practice, $\boldsymbol{\phi}$ will be compactly supported; more generally, $\boldsymbol{\phi}$ should have good (exponential) decay, so that $\boldsymbol{\phi}$ will be smooth. This means that we expect that in

$$\hat{\boldsymbol{\phi}}(u) = \boldsymbol{P}\left(\frac{u}{2}\right) \cdots \boldsymbol{P}\left(\frac{u}{2^n}\right) \hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right) = \boldsymbol{P}\left(\frac{u}{2}\right) \cdots \boldsymbol{P}\left(\frac{u}{2^n}\right) \boldsymbol{a} + \boldsymbol{P}\left(\tfrac{u}{2}\right) \cdots \boldsymbol{P}\left(\tfrac{u}{2^n}\right) \left[\hat{\boldsymbol{\phi}}\left(\tfrac{u}{2^n}\right) - \hat{\boldsymbol{\phi}}(0)\right] \ ,$$

the second term should become negligibly small in the limit for $n \to \infty$. This suggests that we <u>define</u>

$$\hat{\boldsymbol{\Upsilon}}_n(u) := \boldsymbol{P}\left(\frac{u}{2}\right) \cdots \boldsymbol{P}\left(\frac{u}{2^n}\right) \boldsymbol{a} = \boldsymbol{\Pi}_n(u)\boldsymbol{a}$$

and study its limit for $n \to \infty$. In this section, we shall discuss the existence of this limit, pointwise in u. In what follows, $\|\boldsymbol{v}\|$ will denote the Euclidean norm of $v \in \mathbb{R}^d$, i.e., $\|\boldsymbol{v}\| = [v_0^2 + \cdots + v_{r-1}^2]^{1/2}$, and $\|\boldsymbol{v}\| = max \|\boldsymbol{V}\boldsymbol{v}\|/\|\boldsymbol{v}\|$ will be the corresponding matrix norm for $\boldsymbol{V} \in \mathbb{R}^{r\times r}$.

Lemma 3.1 *Suppose that $\boldsymbol{a}$ is an eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. Further, suppose that P is Hölder continuous in $u = 0$,*

$$\|\boldsymbol{P}(u) - \boldsymbol{P}(0)\| \leq C\,|u|^\alpha \ , \tag{3.1}$$

for some $\alpha > 0$, and that

$$\|\boldsymbol{P}(0)\| \leq 2^\alpha \ .$$

Then the infinite product

$$\hat{\boldsymbol{\Upsilon}}(u) := \lim_{n\to\infty} \boldsymbol{\Pi}_n(u)\,\boldsymbol{a} \tag{3.2}$$

converges pointwise for any $u \in \mathbb{R}$. The convergence is uniform on compact sets.

Proof: The Hölder continuity (3.1) implies that

$$\|\boldsymbol{P}(u)\| \leq \|\boldsymbol{P}(0)\| + C\,|u|^\alpha \leq \|\boldsymbol{P}(0)\|\, e^{C'|u|^\alpha}.$$

Hence, we have

$$\begin{aligned} \|\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^l}\right)\| &\leq e^{C'|u|^\alpha[2^{-\alpha}+\ldots+2^{-l\alpha}]}\,\|\boldsymbol{P}(0)\|^l \\ &\leq e^{C_\alpha|u|^\alpha}\,\|\boldsymbol{P}(0)\|^l \end{aligned}$$

since $[2^{-\alpha}+\ldots+2^{-l\alpha}] \leq \frac{2^{-\alpha}}{1-2^\alpha} < \infty$ for $\alpha > 0$. Using this estimation and observing that

$$\begin{aligned} \boldsymbol{\Pi}_k(u)\,\boldsymbol{a} - \boldsymbol{a} &= \left[\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^k}\right) - \boldsymbol{P}(0)^k\right]\,\boldsymbol{a} \\ &= \sum_{l=1}^{k}\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^{l-1}}\right)\,\left[\boldsymbol{P}\left(\frac{u}{2^l}\right) - \boldsymbol{P}(0)\right]\,\boldsymbol{P}(0)^{k-l}\,\boldsymbol{a}\,, \end{aligned} \tag{3.3}$$

it follows that for any $k \in \mathbb{N}$

$$\begin{aligned} \|\boldsymbol{\Pi}_k(u)\,\boldsymbol{a} - \boldsymbol{a}\| &\leq C\,e^{C_\alpha|u|^\alpha}\,|u|^\alpha\sum_{l=1}^{\infty}\left[\frac{\|\boldsymbol{P}(0)\|}{2^\alpha}\right]^l \\ &\leq C'\,e^{C_\alpha|u|^\alpha}\,|u|^\alpha\,, \end{aligned}$$

where we assume that a is normalized, $\|a\| = 1$, for the sake of convenience. Now, considering the Cauchy sequence, we have

$$\|\boldsymbol{\Pi}_{N+k}(u)\,\boldsymbol{a} - \boldsymbol{\Pi}_N(u)\boldsymbol{a}\| \leq C'\,e^{C_\alpha|u|^\alpha(1+2^{-N\alpha})}\,|u|^\alpha\left(\frac{\|\boldsymbol{P}(0)\|}{2^\alpha}\right)^N,$$

i.e.,

$$\lim_{N\to\infty}\|\boldsymbol{\Pi}_{N+k}(u)\,\boldsymbol{a} - \boldsymbol{\Pi}_N(u)\,\boldsymbol{a}\| = 0.$$

Thus, (3.2) converges pointwise for all $u \in \mathbb{R}$. The convergence is uniform on compact sets. ■

This result is often sufficient. The argument can be pushed a little further, allowing for the substitution of $\|\boldsymbol{P}(0)\|$ by the spectral radius of $\boldsymbol{P}(0)$,

$$\rho_0 = \rho(\boldsymbol{P}(0)) := \max\{|\lambda| : \boldsymbol{P}(0)\,\boldsymbol{x} = \lambda\,\boldsymbol{x},\ \boldsymbol{x} \neq \boldsymbol{0}\}\,.$$

Theorem 3.2 *Let $\boldsymbol{a}$ be an eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. Suppose that $\boldsymbol{P}(u)$ is Hölder continuous in $u = 0$ with Hölder exponent α, and that*

$$\rho_0 < 2^\alpha\,. \tag{3.4}$$

Then $\hat{\boldsymbol{\Upsilon}}(u)$, defined by (3.2), converges pointwise for all $u \in \mathbb{R}$, and the convergence is uniform on compact sets. Moreover, $\hat{\boldsymbol{\Upsilon}}(u)$ is Hölder continuous in $u = 0$.

Proof: 1. Again, we assume $\|a\| = 1$ for the sake of convenience. Let $\boldsymbol{Q}(u) := \boldsymbol{P}(u) - \boldsymbol{P}(0)$. Then it follows by Hölder continuity that $\|\boldsymbol{Q}(u)\| \le C\,|u|^\alpha$ with $\alpha \ge 1$. Further, observe that $\|\boldsymbol{P}(0)^k\| \le C_\epsilon\,(\rho_0+\epsilon)^k$. Then we have

$$\begin{aligned}
\|\boldsymbol{\Pi}_N(u)\| &= \|\boldsymbol{P}\left(\frac{u}{2}\right)\dots\boldsymbol{P}\left(\frac{u}{2^N}\right)\| \\
&= \|[\boldsymbol{P}(0)+\boldsymbol{Q}\left(\frac{u}{2}\right)]\dots[\boldsymbol{P}(0)+\boldsymbol{Q}\left(\frac{u}{2^N}\right)]\| \\
&= \|\sum_{l=0}^{N}\sum_{\substack{m_1,m_2,\dots,m_{l+1}=0\\ m_1+m_2+\dots+m_{l+1}=N-l}}^{N-l} \boldsymbol{P}(0)^{m_1}\,\boldsymbol{Q}\left(\frac{u}{2^{m_1+1}}\right)\,\boldsymbol{P}(0)^{m_2}\,\boldsymbol{Q}\left(\frac{u}{2^{m_1+m_2+2}}\right)\dots \\
&\quad \times\boldsymbol{Q}\left(\frac{u}{2^{m_1+m_2+\dots+m_l+l}}\right)\,\boldsymbol{P}(0)^{m_{l+1}}\|.
\end{aligned}$$

Introducing $b = 2^{-\alpha} < 1$, this leads to

$$\begin{aligned}
\|\boldsymbol{\Pi}_N(u)\| \le \sum_{l=0}^{N}\sum_{\substack{m_1,m_2,\dots,m_{l+1}=0\\ m_1+m_2+\dots+m_{l+1}=N-l}}^{N-l} C_\epsilon^{l+1}\,(\rho_0+\epsilon)^{N-l}\,C^l\,|u|^{\alpha l}\,2^{-\alpha\sum_{k=0}^{l}(m_k+1)(l+1-k)} \\
\le\ C_\epsilon b^{-N+l}\,b^{l(l+1)/2}\sum_{l=0}^{N}(\rho_0+\epsilon)^{N-l}\,(|u|^\alpha C_\epsilon C)^l \sum_{\substack{m_1,m_2,\dots,m_{l+1}=0\\ m_1+m_2+\dots+m_{l+1}=N-l}}^{N-l} b^{[m_1(l+1)+\dots+m_{l+1}]}
\end{aligned}$$

2. Next we find an upper bound for the sum over $m_1,\dots,m_{l+1}$. Consider the sum

$$A_{M,L} := \sum_{\substack{m_1,m_2\dots,m_L=0\\ m_1+m_2+\dots+m_L=M}}^{M} b^{m_1+2m_2+\dots Lm_L} \tag{3.5}$$

For $A_{M,L}$ we find the recursion (putting $m = m_1$)

$$A_{M,L} = \sum_{m=0}^{M} b^M\,A_{M-m,L-1} = b^M\,\sum_{m=0}^{M} A_{M-m,L-1}$$

with $A_{M,1} = b^M$ and $A_{0,L} = 1$. We show by induction that

$$A_{M,L} \le \frac{b^M}{(1-b)^{L-1}}. \tag{3.6}$$

For $L = 1$ and $M \in \mathbb{N}$, (3.6) is satisfied. Now, assume that (3.6) holds for $L \ge 1$ and $M \in \mathbb{N}$. Then we obtain by the recursion formula

$$A_{M,L+1} = b^M\sum_{m=0}^{M} A_{M-m,L} \le \frac{b^M}{(1-b)^{L-1}}\sum_{m=0}^{M} b^m \le \frac{b^M}{(1-b)^L}.$$

3. Substituting (3.6) into the expression for $\|\mathbf{\Pi}_N(u)\|$ obtained above, we find

$$\begin{aligned}\|\mathbf{\Pi}_N(u)\| &\leq C_\epsilon \sum_{l=0}^{N} (\rho_0+\epsilon)^{N-l} \, (|u|^\alpha \, C_\epsilon C)^l \, b^{-N+l} \, b^{l(l+1)/2} \, A_{N-l,l+1} \\ &\leq C_\epsilon \sum_{l=0}^{N} (\rho_0+\epsilon)^{N-l} \, (|u|^\alpha \, C_\epsilon C)^l \, \frac{b^{l(l+1)/2}}{(1-b)^l} \\ &\leq C_\epsilon \, (\rho_0+\epsilon)^N \sum_{l=0}^{N} \left[\frac{|u|^\alpha \, C_\epsilon \, C}{(\rho_0+\epsilon)(1-b)} \right]^l b^{l(l+1)/2}. \end{aligned} \tag{3.7}$$

The sum in (3.7) converges uniformly for $|u| \leq \Omega$, since $b < 1$. Hence, we can estimate

$$\|\mathbf{\Pi}_N(u)\| \leq C_{\epsilon,\Omega} \, (\rho_0+\epsilon)^N. \tag{3.8}$$

3. Now, with the same argument as in the proof of Lemma 3.1, we have by (3.3)

$$\begin{aligned}\|\mathbf{\Pi}_k(u) \, \boldsymbol{a} - \boldsymbol{a}\| &\leq C \sum_{l=1}^{k} C_{\epsilon,\Omega} \, (\rho_0+\epsilon)^{l-1} \left(\frac{|u|}{2^l} \right)^\alpha \\ &\leq C'_{\epsilon,\Omega} \, |u|^\alpha \sum_{l=1}^{k} \left(\frac{\rho_0+\epsilon}{2^\alpha} \right)^l. \end{aligned} \tag{3.9}$$

Hence, convergence of $\|\mathbf{\Pi}_k(u)\boldsymbol{a} - \boldsymbol{a}\|$ for any $k \in \mathbb{N}$ is ensured, if $\rho_0 < 2^\alpha$, by choosing ϵ sufficiently small. Again, it follows that

$$\begin{aligned}\|\mathbf{\Pi}_{N+k}(u)\boldsymbol{a} - \mathbf{\Pi}_N(u)\boldsymbol{a}\| &\leq C_{\epsilon,\Omega} \, C'_{\epsilon,\omega} (\rho_0+\epsilon)^N \, \frac{|u|^\alpha}{2^{N\alpha}} \sum_{l=1}^{k} \left(\frac{\rho_0+\epsilon}{2^\alpha} \right)^l \\ &\leq C''_{\epsilon,\Omega} \, |u|^\alpha \left(\frac{\rho_0+\epsilon}{2^\alpha} \right)^N, \end{aligned}$$

where the last term is uniformly small in k if N is sufficiently large. Thus, for fixed u, $\mathbf{\Pi}_k(u)\boldsymbol{a}$ is a Cauchy sequence for $\rho_0 < 2^\alpha$, implying that we have pointwise convergence of (3.2). Moreover, the convergence is uniform on compact sets. The Hölder continuity of (3.1) in $u = 0$ directly follows from (3.9). ∎

4. Decay of infinite matrix products

Having shown that $\hat{\mathbf{\Upsilon}}(u)$ is well-defined (under some conditions on $\boldsymbol{P}(u)$), we now proceed to study how the factorization (2.8) of the refinement mask $\boldsymbol{P}(u)$ can lead to decay in u of $\hat{\mathbf{\Upsilon}}(u)$ for $|u| \to \infty$. Let us suppose that $\boldsymbol{P}(u)$ can be factored in the form

$$\boldsymbol{P}(u) = \frac{1}{2^m (1-e^{-iu})^m} \, \boldsymbol{C}_{\boldsymbol{x}_0}(2u) \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u) \, \boldsymbol{P}^{(m)}(u) \, \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(u) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}(u),$$

where the $\boldsymbol{C}$– and $\boldsymbol{E}$–matrices are defined as in (2.2) and (2.4) and where the vectors $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1}$ are all different from zero. We can now rewrite $\hat{\Upsilon}(u)$ as

$$\begin{aligned}\hat{\Upsilon}(u) \quad := \quad & \lim_{n\to\infty} \boldsymbol{\Pi}_n(u)\,\boldsymbol{a} = \lim_{n\to\infty} \left\{ \left[\frac{1}{2^n(1-e^{-iu/2^n})} \right]^m \boldsymbol{C}_{\boldsymbol{x}_0}(u) \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u) \right. \\ & \left. \times \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right) \boldsymbol{a} \right\}.\end{aligned}$$

We note again that, since $\boldsymbol{x}_0^T\,\boldsymbol{P}(0) = \boldsymbol{x}_0^T$ with $\boldsymbol{x}_0 \neq 0$, 1 is an eigenvalue of $\boldsymbol{P}(0)$, and we take $\boldsymbol{a}$ to be a right eigenvector of $\boldsymbol{P}(0)$ for that eigenvalue. We also assume that $\boldsymbol{x}_0^T\,\boldsymbol{a} \neq 0$; if the eigenvalue 1 of $\boldsymbol{P}(0)$ is nondegenerate, then this is automatically satisfied. Note that for $u = 0$ we have $\hat{\Upsilon}(0) = \lim_{n\to\infty} \boldsymbol{P}(0)^n\,\boldsymbol{a} = \boldsymbol{a}$. We will establish conditions under which

$$\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right) \boldsymbol{a}\|$$

tends to a finite limit for $n \to \infty$; since $\lim_{n\to\infty} |2^{-n}\,(1 - e^{-iu/2^n})^{-n}| = |u|^{-1}$ for $u \neq 0$, this then implies

$$\begin{aligned}\|\hat{\Upsilon}(u)\| \quad \leq \quad & (1+|u|)^{-m}\,\|\boldsymbol{C}_{\boldsymbol{x}_0}(u) \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)\| \\ & \times \lim_{n\to\infty} \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right) \boldsymbol{a}\|.\end{aligned} \tag{4.1}$$

Let us define the vectors $\boldsymbol{e}_k$ by

$$\begin{aligned}\boldsymbol{e}_{k,\nu} = 1 \quad & if \quad x_{k,\nu} \neq 0 \\ 0 \quad & if \quad x_{k,\nu} = 0\end{aligned} \tag{4.2}$$

where $x_{k,\nu}$ are the components of the vectors $\boldsymbol{x}_k$ $(k = 0, \ldots, m-1)$ introduced above.

Theorem 4.1 *Let $\boldsymbol{P}$ be an $r \times r$–matrix of the form*

$$\boldsymbol{P}(u) = \frac{1}{2^m}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u) \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\,\boldsymbol{P}^{(m)}(u)\,\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)^{-1} \ldots \boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1},$$

where the matrices $\boldsymbol{C}_{\boldsymbol{x}_k}$ are defined by the vectors $\boldsymbol{x}_k \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$ via (2.2) *and where $\boldsymbol{P}^{(m)}(u)$ is an $(r \times r)$ matrix with trigonometric polynomials as entries. Suppose that $\boldsymbol{P}^{(m)}(0)\boldsymbol{e}_{m-1} = \boldsymbol{e}_{m-1}$ where $\boldsymbol{e}_{m-1}$ is defined by* 4.3). Further, suppose that

$$\rho_m := \rho(\boldsymbol{P}^{(m)}(0)) < 2 \tag{4.3}$$

and let

$$\gamma_k := \frac{1}{k}\,\log_2 \sup_u \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^k}\right)\|. \tag{4.4}$$

Then there exists a constant $C > 0$ such that for all $u \in \mathbb{R}$

$$\|\hat{\Upsilon}(u)\| \leq C\,(1+|u|)^{-m+\gamma_k}. \tag{4.5}$$

Note that the requirement that $\boldsymbol{P}^{(m)}\boldsymbol{e}_{m-1} = \boldsymbol{e}_{m-1}$ is automatically satisfied in the case of interest to us, i.e., when $\boldsymbol{P}(u)$ is the refinement mask for functions $\phi_0(x), \ldots, \phi_{r-1}(x)$ whose integer translates provide approximation order m; see Plonka (1995).

Proof: 1. From (2.6) it follows that

$$\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(u) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}(u) = \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}(0) + \sum_{k=1}^{m} (1 - e^{-iu})^k \, \boldsymbol{G}^{(k)}_{\boldsymbol{x}_{m-1},\ldots,\boldsymbol{x}_0}$$

with some matrices $\boldsymbol{G}^{(k)}_{\boldsymbol{x}_{m-1},\ldots,\boldsymbol{x}_0}$ depending on $\boldsymbol{E}_{\boldsymbol{x}_\nu}(0)$ and $(\mathrm{D}\boldsymbol{E}_{\boldsymbol{x}_\nu})(0)$ $(\nu = 0, \ldots, m-1)$. Hence, we can write

$$\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right) \boldsymbol{a} = \boldsymbol{T}_{1,n}(u) + \boldsymbol{T}_{2,n}(u)$$

with

$$\boldsymbol{T}_{1,n}(u) := \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}(0)\, \boldsymbol{a}$$

and

$$\boldsymbol{T}_{2,n}(u) := \sum_{k=1}^{m} (1 - e^{-iu/2^n})^k \, \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \boldsymbol{v}_k$$

where $\boldsymbol{v}_k := \boldsymbol{G}^{(k)}_{\boldsymbol{x}_{m-1},\ldots,\boldsymbol{x}_0}\, \boldsymbol{a}$ $(k = 1, \ldots, m)$.

2. We can estimate the second term $\boldsymbol{T}_{2,n}(u)$ by (3.8),

$$\begin{aligned} \|\boldsymbol{T}_{2,n}(u)\| &\leq C \sum_{k=1}^{m} |2^{-n}u|^k \, \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\| \, \|\boldsymbol{v}_k\| \\ &\leq C_{|u|}\, 2^{-n} \, \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\| \\ &\leq C_{\epsilon,|u|}\, 2^{-n} \, (\rho_m + \epsilon)^n. \end{aligned}$$

Since the spectral radius ρ_m of $\boldsymbol{P}^{(m)}(0)$ is supposed to be < 2, it follows that for all $u \in \mathbb{R}$

$$\lim_{n\to\infty} \|\boldsymbol{T}_{2,n}(u)\| = 0.$$

3. We now concentrate on $\boldsymbol{T}_{1,n}(u)$. From the structure of $\boldsymbol{E}_{\boldsymbol{x}_k}$ and the definition (4.2) of $\boldsymbol{e}_k$ it follows that, for any vector $\boldsymbol{b}$

$$\boldsymbol{E}_{\boldsymbol{x}_k}(0)\, \boldsymbol{b} = (\boldsymbol{x}_k)^T \boldsymbol{b}\, \boldsymbol{e}_k \quad (k = 0, \ldots, m-1).$$

Repeating this argument, we obtain

$$\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0) \ldots \boldsymbol{E}_{\boldsymbol{x}_0}(0)\, \boldsymbol{a} = [(\boldsymbol{x}_0)^T \boldsymbol{a}]\, [(\boldsymbol{x}_1)^T \boldsymbol{e}_0] \ldots [(\boldsymbol{x}_{m-1})^T \boldsymbol{e}_{m-2}]\, \boldsymbol{e}_{m-1}.$$

This leads to

$$\boldsymbol{T}_{1,n}(u) = [(\boldsymbol{x}_0)^T \boldsymbol{a}]\, [(\boldsymbol{x}_1)^T \boldsymbol{e}_0] \ldots [(\boldsymbol{x}_{m-1})^T \boldsymbol{e}_{m-2}]\, \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \boldsymbol{e}_{m-1}.$$

Since $\boldsymbol{P}^{(m)}(0)\,\boldsymbol{e}_{m-1} = \boldsymbol{e}_{m-1}$, and $\rho_m < 2$, we find by Theorem 3.2 that $\lim_{n\to\infty} \boldsymbol{T}_{1,n}(u)$ is well–defined for all u, and uniformly bounded on compact sets.
4. Take now any $u \in \mathbb{R}$. If $|u| \leq 1$ then there is a C such that $\|\boldsymbol{T}_{1,n}(u)\| \leq C$. If $|u| > 1$, define L such that $2^{L-1} < |u| \leq 2^L$. Thus,

$$\begin{aligned}\| \lim_{n\to\infty} \boldsymbol{T}_{1,n}(u)\| &\leq \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^L}\right)\| \, \| \lim_{n\to\infty} \boldsymbol{T}_{1,n}\left(\frac{u}{2^L}\right)\| \\ &\leq C\, \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^L}\right)\|.\end{aligned}$$

By the definition of γ_k it follows

$$\| \lim_{n\to\infty} \boldsymbol{T}_{1,n}(u)\| \leq C'\, 2^{L\,\gamma_k} \leq C''\,(1+|u|)^{\gamma_k},$$

i.e., by (4.1) and the observations above we find a constant C such that

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \leq C\,(1+|u|)^{-m+\gamma_k}. \quad \blacksquare$$

Remarks.
1. It follows from (4.5) that the components of $\boldsymbol{\Upsilon}(x)$ are continuous if $\boldsymbol{P}$ satisfies the above conditions and if $\gamma_k < m-1$.
2. For the proof of Theorem 4.1 we have assumed that $\rho(\boldsymbol{P}^{(m)}(0)) < 2$. As we will see in Lemma 4.3 below, this can be ensured if the largest eigenvalue of $\boldsymbol{P}(0)$ apart from the eigenvalue 1 is smaller than 2^{-m+1}.
3. In order to avoid that $\boldsymbol{T}_{1,n}(u)$ collapses to 0 as $n \to \infty$, i.e., $\lim_{n\to\infty} \|\boldsymbol{T}_{1,n}(u)\| = 0$, which would imply $\hat{\boldsymbol{\Upsilon}}(u) = 0$, we have to make sure that

$$[(\boldsymbol{x}_0)^T \boldsymbol{a}]\,[(\boldsymbol{x}_1)^T \boldsymbol{e}_0] \ldots [(\boldsymbol{x}_{m-1})^T \boldsymbol{e}_{m-2}] \neq 0. \tag{4.6}$$

Note that this is already satisfied if there is an index ν $(0 \leq \nu \leq r-1)$ such that the νth component of $\boldsymbol{x}_k$ does not vanish for all $k = 0, \ldots, m-1$. On the other hand, since $\boldsymbol{x}_l$ is a left eigenvector, and $\boldsymbol{e}_{l-1}$ a right eigenvector of $\boldsymbol{P}^{(l)}(0)$, both for the eigenvalue 1, (4.6) is also satisfied if the eigenvalue 1 of $\boldsymbol{P}^{(l)}(0)$ is nondegerate, for all l, or equivalently, if the eigenvalues $1, 1/2, \ldots, 2^{-m+1}$ of $\boldsymbol{P}(0)$ are all nondegenerate (see Lemma 4.3 below).

More detailed estimates show that decay of $\hat{\boldsymbol{\Upsilon}}(u)$ is also possible in some cases where $\rho(\boldsymbol{P}^{(m)}) \geq 2$.

Corollary 4.2 *Let $\boldsymbol{P}$ be again an $r \times r$–matrix of the form*

$$\boldsymbol{P}(u) := \frac{1}{2^m}\, \boldsymbol{C}_{\boldsymbol{x}_0}(2u) \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\, \boldsymbol{P}^{(m)}(u)\, \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)^{-1} \ldots \boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1},$$

where the matrices $\boldsymbol{C}_{\boldsymbol{x}_k}$ *are defined by the vectors* $\boldsymbol{x}_k \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$ *via* (2.2) *and where* $\boldsymbol{P}^{(m)}(u)$ *is an* $(r \times r)$ *matrix with trigonometric polynomials as entries. Suppose that* $P^m(0)\ell_{m-1} = \ell_{m-1}$. *Further, suppose that*

$$\|\boldsymbol{P}^{(m)}(u) - \boldsymbol{P}^{(m)}(0)\| \leq C\,|u|^{\alpha} \tag{4.7}$$

and that the $\boldsymbol{E}$*-matrices defined in* (2.4) *satisfy*

$$\|\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(u)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(u) - \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(0)\| \leq C\,|u|^{\beta}. \tag{4.8}$$

Now, if $\rho_m < 2^{\min\{\alpha,\beta\}}$, *then there exists a constant* $C > 0$ *such that for all* $u \in \mathbb{R}$

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \leq C\,(1 + |u|)^{-m+\gamma_k},$$

where γ_k *is defined in* (4.4).

Proof: Observe that

$$\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\,\boldsymbol{a}\| \leq \|\boldsymbol{S}_n(u)\| + \|\boldsymbol{T}_{1,n}\|,$$

with

$$\begin{aligned}\boldsymbol{S}_n(u) \;&:=\; \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\\ &\quad\times\left[\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right) - \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(0)\right]\boldsymbol{a}\end{aligned}$$

and where $\boldsymbol{T}_{1,n}$ is defined as in the proof of Theorem 4.1. With the same argument as in Theorem 3.2 (cf. (3.8)) we obtain by (4.8) that

$$\begin{aligned}\|\boldsymbol{S}_n(u)\| \;&\leq\; \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\|\\ &\qquad[\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right) - \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(0)]\,\boldsymbol{a}\|\\ &\leq\; C_{\epsilon,\Omega}\,(\rho_m + \epsilon)^n\,C\,\frac{|u|^{\beta}}{2^{n\beta}}.\end{aligned}$$

Thus $\boldsymbol{S}_n(u)$ tends to zero for $n \to \infty$ if $\rho_m < 2^{\beta}$. Further, since $\boldsymbol{e}_{m-1}$ is an eigenvector of $\boldsymbol{P}^{(m)}(0)$, we can apply Lemma 3.1 in order to show that $\boldsymbol{T}_{1,n}$ is convergent for $(\rho_m + \epsilon) < 2^{\alpha}$. Hence,

$$\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\,\boldsymbol{a}$$

is well–defined if $\rho_m < 2^{\min\{\alpha,\beta\}}$. Following point 4 of the proof of Theorem 4.1 we can find a constant C such that

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \leq C\,(1 + |u|)^{-m+\gamma_k}. \qquad \blacksquare$$

Since $\boldsymbol{P}^{(m)}(u)$ is completely determined by $\boldsymbol{P}(u)$, the conditions $\rho_m < 2$ or $\rho_m < 2^{\gamma}$ are restrictions on $\boldsymbol{P}(u)$. The following lemma shows that there is a simple connection between the spectra of $\boldsymbol{P}(0)$ and $\boldsymbol{P}^{(m)}(0)$, which makes it possible to recast bounds on ρ_m as spectral bounds on $\boldsymbol{P}(0)$ as well.

Lemma 4.3 *Let $\boldsymbol{P}(u)$ be an $r \times r$–matrix of the form*

$$\boldsymbol{P}(u) = \frac{1}{2}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\,\boldsymbol{P}^{(1)}(u)\,\boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1}, \tag{4.9}$$

where $\boldsymbol{C}_{\boldsymbol{x}_0}$ is defined by $\boldsymbol{x}_0 \neq \boldsymbol{0}$ via (2.2), and assume that $\boldsymbol{e}_0$ (defined by $\boldsymbol{x}_0$ via (4.2)) is the right eigenvector of $\boldsymbol{P}^{(1)}(0)$ for the eigenvalue 1. Further, let $\{1, \mu_1, \ldots, \mu_{r-1}\}$ be the spectrum of $\boldsymbol{P}(0)$. Then the spectrum of $\boldsymbol{P}^{(1)}(0)$ is $\{1, 2\mu_1, \ldots, 2\mu_{r-1}\}$.

Proof: 1. First, observe that the factorization (4.9) implies that $\boldsymbol{P}(0)$ has the eigenvalue 1 with left eigenvector $\boldsymbol{x}_0$. At the same time, $\boldsymbol{x}_0$ is a left eigenvector of $\boldsymbol{P}(\pi)$ for the eigenvalue 0, i.e., we have

$$(\boldsymbol{x}_0)^T\,\boldsymbol{P}(0) = \boldsymbol{x}_0, \qquad (\boldsymbol{x}_0)^T\,\boldsymbol{P}(\pi) = \boldsymbol{0}$$

(cf. Plonka (1995), Theorem 4.1).
2. Without loss of generality, we assume that $\boldsymbol{x}_0$ is of the form $\boldsymbol{x}_0 = (x_{0,0}, \ldots, x_{0,l-1}, 0, \ldots, 0)^T$ with $1 \leq l \leq r$ and $x_{0,\nu} \neq 0$ for $\nu = 0, \ldots, l-1$. Now, consider $\boldsymbol{P}^\sharp(u) := \boldsymbol{M}\,\boldsymbol{P}(u)\,\boldsymbol{M}^{-1}$ with

$$\boldsymbol{M} := \begin{pmatrix} x_{0,0} & x_{0,1} & \ldots & x_{0,l-1} \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \oplus \boldsymbol{I}_{r-l},$$

where $\boldsymbol{I}_{r-l}$ is the $(r-l) \times (r-l)$ unit matrix (cf. section 2). Since $\boldsymbol{M}$ is invertible, $\boldsymbol{P}^\sharp(0)$ possesses the same spectrum as $\boldsymbol{P}(0)$. Now, the left eigenvector of $\boldsymbol{P}^\sharp(0)$ for the eigenvalue 1 is $\boldsymbol{x}^\sharp := (\boldsymbol{x}_0)^T\,M^{-1} = (1, 0, \ldots, 0)^T$. Analogously, we obtain that $\boldsymbol{x}^\sharp$ is the left eigenvector of $\boldsymbol{P}^\sharp(\pi)$ for the eigenvalue 0. Hence, we find the factorization

$$\boldsymbol{P}^\sharp(u) = \frac{1}{2}\,\boldsymbol{C}^\sharp(2u)\,\boldsymbol{P}^{\sharp\,(1)}(u)\,\boldsymbol{C}^\sharp(u)^{-1} \tag{4.10}$$

with

$$\boldsymbol{C}^\sharp(u) := \text{diag}\,(1 - e^{-iu},\, 1, \ldots, 1).$$

Observing that $\boldsymbol{P}^\sharp(0)$ has the structure

$$\boldsymbol{P}^\sharp(0) = \begin{pmatrix} 1 & 0 \ldots 0 \\ \boxed{\boldsymbol{r}(0)} & \boxed{\boldsymbol{R}(0)} \end{pmatrix},$$

where $\boldsymbol{r}(0)$ is a vector of length $r-1$ and $\boldsymbol{R}(0)$ an $(r-1) \times (r-1)$–matrix, it follows by the factorization (4.10) that $\boldsymbol{P}^{\sharp\,(1)}(0)$ is of the form

$$\boldsymbol{P}^{\sharp\,(1)}(0) = \begin{pmatrix} 1 & 0 \ldots 0 \\ \boxed{\boldsymbol{0}} & \boxed{2\,\boldsymbol{R}(0)} \end{pmatrix}.$$

Consequently, $\boldsymbol{P}^{\sharp\,(1)}(0)$ has the eigenvalues $1, 2\mu_1, \ldots, 2\mu_{r-1}$.
3. We show next that $\boldsymbol{P}^{\sharp\,(1)}(0)$ and $\boldsymbol{P}^{(1)}(0)$ have the same spectrum. The factorizations (4.9) and (4.10) imply the following connection between $\boldsymbol{P}^{\sharp\,(1)}(0)$ and $\boldsymbol{P}^{(1)}(0)$:

$$\boldsymbol{P}^{(1)}(u) = \boldsymbol{A}(2u)\,\boldsymbol{P}^{\sharp\,(1)}(u)\,\boldsymbol{A}(u)^{-1}(u)\ ,$$

where

$$\begin{aligned}\boldsymbol{A}(u) &:= \boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1}\,\boldsymbol{M}^{-1}\,\boldsymbol{C}^{\sharp}(u)\\ &= \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ z & x_{0,1} & x_{0,2} & \ldots & x_{0,r-1} \\ z & 0 & x_{0,2} & \ldots & x_{0,r-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ z & 0 & \ldots & 0 & x_{0,r-1} \end{pmatrix} \oplus \boldsymbol{I}_{r-l} \qquad (z := e^{-iu}).\end{aligned}$$

Since $\boldsymbol{A}(0)$ is invertible, it follows that $\boldsymbol{P}^{\sharp\,(1)}(0)$ and $\boldsymbol{P}^{(1)}(0)$ are similar, and thus $\boldsymbol{P}^{(1)}(0)$ has spectrum $\{1,\, 2\,\mu_1, \ldots 2\,\mu_{r-1}\}$. ∎

It follows that the spectrum of $\boldsymbol{P}^{(m)}(0)$ is likewise given by $\{1, 2^m\mu_1, \ldots, 2^m\mu_{r-1}\}$. The requirement that $\rho_m < 2^\lambda$ (as in Corollary (4.4) thus translates into

$$max\{|\mu_1|, \ldots, |\mu_{r-1}|\} < 2^{\lambda-m}\ .$$

Remark.
If $m > \lambda$, which need not be true in general, but which we expect to be true in most cases ($\lambda = 1$ except if both $\boldsymbol{P}(u)$ and $\boldsymbol{E}_{x_{m-1}}(u)\boldsymbol{E}_{x_0}(u)$ have Hölder exponents > 1 in $u = 0$; since $\gamma_k \geq 0$ for all k, we have $m \geq 2$ in order to ensure decay faster than $(1+|u|)^{-1-\epsilon}$ for $|\hat{\boldsymbol{\Upsilon}}(u)|$), then (4.10) automatically implies that ρ_0, the spectral radius of $\boldsymbol{P}(0)$, equals 1. It also implies that the eigenvalue 1 of $\boldsymbol{P}(0)$ is nondegenerate.

5. Uniqueness

If the conditions of Theorem (4.1) are satisfied, with $\gamma_k < m-1$ for some $k \leq 1$, then $\hat{\boldsymbol{\Upsilon}}$ is well-defined and integrable, so that $\boldsymbol{\Upsilon}(x)$, its inverse Fourier transform, is well-defined as well. Since $\hat{\boldsymbol{\Upsilon}}(u)$ is obviously a solution to (1.6), $\boldsymbol{\Upsilon}(x)$ is a solution to (1.1). Is it the only one? The following theorem lists some conditions that ensure uniqueness.

Theorem 5.1 *Suppose that the conditions of Theorem* 4.1 *are satisfied, with* $\inf_{k\geq 1}\gamma_k < m-1$, *and that the eigenvalue* 1 *of* $\boldsymbol{P}(0)$ *is nondegenerate. Then* $\boldsymbol{\Upsilon}(x)$ *is a compactly supported continuous solution to* (1.1). *Moreover, if* $\boldsymbol{\phi}(x)$ *is any other* L^1*-solution to* (1.1) *such that* $\int \boldsymbol{\phi}(x)dx \neq \mathbf{0}$ *and* $\int(1+|x|)\|\boldsymbol{\phi}(x)\|dx < \infty$, *then* $\boldsymbol{\phi}(x)$ *is a multiple of* $\boldsymbol{\Upsilon}(x)$.

Proof: 1. We are assuming, as in Theorem 4.1, that all the entries of $\boldsymbol{P}(u)$ are trigonometric polynomials. Let us, for this point only, consider u to be complex rather than real. The argument that $\|\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\|$ is bounded uniformly in $n \geq 1$ and in $u \in \Delta(0,1) = \{z; |z| < 1\}$ holds for complex u as well. Since $\|\boldsymbol{P}(u)\| \leq Ce^{R|Imu|}$, it then follows that for any $|u| > 1$, $2^k \leq |u| < 2^{k+1}$, $\|\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}\| \leq C^k e^{R|Imu|(1/2+1/4+\ldots+1/2^k)}C_1 \leq C'(1+|u|)^k e^{R|Imu|}$. It follows that $\hat{\boldsymbol{\Upsilon}}(u) = \lim_{n\to\infty}\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}$ satisfies the same bound, implying that $\boldsymbol{\Upsilon}$ is a compactly supported distribution. On the other hand, $\boldsymbol{\Upsilon}(x)$ is bounded and continuous because, by Theorem (4.1), $|\hat{\boldsymbol{\Upsilon}}(u)| \leq C(1+|u|)^{-1-\epsilon}$ for real u.

2. If $\boldsymbol{\phi}(x)$ is another L^1-solution, then $\hat{\boldsymbol{\phi}}(0) \neq 0$ must be an eigenvector for $\boldsymbol{P}(0)$ with eigenvalue 1, so that $\hat{\boldsymbol{\phi}}(0) = c\boldsymbol{a}$ for some $c \neq 0$. Since $\int |x|\|\boldsymbol{\phi}(x)\|dx < \infty$, we also have $\|\hat{\boldsymbol{\phi}}(u) - \hat{\boldsymbol{\phi}}(0)\| \leq C|u|$. Hence, for any fixed u,

$$\begin{aligned}
&\|\hat{\boldsymbol{\phi}}(u) - c\hat{\boldsymbol{\Upsilon}}(u)\| \\
&\quad = \lim_{n\to\infty}\|\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\left[\hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right) - \hat{\boldsymbol{\phi}}(0)\right]\| \\
&\quad \leq C'\lim_{n\to\infty}\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\boldsymbol{E}\boldsymbol{x}_{m-1}\left(\frac{u}{2^n}\right)\cdots\boldsymbol{E}\boldsymbol{x}_0\left(\frac{u}{2^n}\right)\left[\hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right) - \hat{\boldsymbol{\phi}}(0)\right]\| \\
&\quad \leq C'\lim_{n\to\infty}\left[C_{\epsilon,|u|}(\rho_m+\epsilon)^m C_1 C|u|2^{-n}\right] \\
&\quad = 0\ ,
\end{aligned}$$

since $\rho_m < 2$. Hence, $\boldsymbol{\phi} \equiv c\boldsymbol{\Upsilon}$. ∎

All the examples studied in the literature so far correspond to $\boldsymbol{P}(u)$ in which all the entries are trigonometric polynomials, and that is why we have mostly restricted ourselves to this case. Nevertheless, most of our analysis carries over to the nonpolynomial case. In Plonka (1995), the original version of Theorem 2.1 does not require the ϕ_ν to be compactly supported, nor the $\boldsymbol{P}_{\nu\rho}(u)$ to be trigonometric polynomials; only sufficient decay in x for $\boldsymbol{\phi}(x)$ and a sufficiently high regularity of $\boldsymbol{P}(u)$ are required. As shown in section 3 (where the $\boldsymbol{P}(u)$ were not restricted to trigonometric polynomials), this then implies $|\hat{\boldsymbol{\Upsilon}}(u) - \hat{\boldsymbol{\Upsilon}}(0)| \leq C|u|$ (since $\boldsymbol{P}$ is Hölder continuous in $u = 0$ with Hölder exponent at least 1). This, in turn, is the only ingredient necessary in point 2 of the proof of Theorem 5.1, which establishes uniqueness of the solution within a certain class of functions with mild decay. Compact support of $\boldsymbol{\Upsilon}(x)$ is, of course, no longer assured.

6. L^2–CONVERGENCE AND POINTWISE CONVERGENCE

If $\boldsymbol{P}(u)$ is m ($m \geq 1$) times factorizable (in the sense of Plonka (1995)), i.e.,

$$\boldsymbol{P}(u) = \frac{1}{2^m(1-e^{-iu})^m}\boldsymbol{C}\boldsymbol{x}_0(2u)\ldots\boldsymbol{C}\boldsymbol{x}_{m-1}(2u)\,\boldsymbol{P}^{(m)}(u)\,\boldsymbol{E}\boldsymbol{x}_{m-1}(u)\ldots\boldsymbol{E}\boldsymbol{x}_0(u),$$

if the spectral radius of $\boldsymbol{P}^{(m)}(0)$ is less than 2, and if, for some fixed k,

$$\gamma_k = \frac{1}{k} \log_2 \sup_u \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \dots \boldsymbol{P}^{(m)}\left(\frac{u}{2^k}\right)\| < m,$$

then our analysis in the previous sections has shown that

$$\hat{\boldsymbol{\Upsilon}}(u) = \lim_{n\to\infty} \boldsymbol{P}\left(\frac{u}{2}\right) \dots \boldsymbol{P}\left(\frac{u}{2^n}\right) \boldsymbol{a}$$

(with $(\boldsymbol{x}_0)^T \boldsymbol{a} = 1$) is well-defined, and that

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \leq C\,(1 + |u|)^{-m+\gamma_k}.$$

Moreover, we have $\|\hat{\boldsymbol{\Upsilon}}(u) - \hat{\boldsymbol{\Upsilon}}(0)\| \leq C\,|u|$ for $|u| \leq 1$. So far, this convergence is only pointwise, in the Fourier domain. How about L^2– or L^1–convergence? (This would then imply L^2–convergence or pointwise convergence in the "x–domain".)

First of all, we have to change the set–up slightly, since $\boldsymbol{P}\left(\frac{u}{2}\right) \dots \boldsymbol{P}\left(\frac{u}{2^n}\right) \boldsymbol{a}$ is $2^{n+1}\pi$–periodic and cannot be in L^2 or L^1. Let now f be a C^2 function with $\hat{f}(0) \neq 0$. Normalize f so that $\hat{f}(0) = 1$. We also assume that

$$\hat{f}(2\pi k) = 0, \qquad (\mathrm{D}\hat{f})(2\pi k) = 0 \qquad (k \in \mathbb{Z} \setminus \{0\}) \tag{6.1}$$

and furthermore that

$$|(\mathrm{D}^2\hat{f})(u)| \;\leq\; C\,(1 + |u|)^{-1-\epsilon}, \tag{6.2}$$

for some $\epsilon > 0$. (Take for instance $\hat{f}(u) = \left(\frac{\sin u/2}{u/2}\right)^2$, i.e., f is the cardinal linear B-spline.) Define now the n-tuple of functions

$$\hat{\boldsymbol{\Phi}}_0(u) := \hat{f}(u)\, \hat{\boldsymbol{\Upsilon}}(0)\,, \tag{6.3}$$

Note that $\hat{\boldsymbol{\Phi}}_0(0) = \hat{\boldsymbol{\Upsilon}}(0)$.

Theorem 6.1 *Let $\boldsymbol{P}$ be an $r \times r$ matrix with the assumptions of Theorem* 4.1. *If γ_k defined in* (4.4) *satisfies $\gamma_k < m - 1$, then we have*

$$\lim_{n\to\infty} \|\hat{\boldsymbol{\Phi}}_n - \hat{\boldsymbol{\Upsilon}}\|_{L^1} = 0,$$

where $\hat{\boldsymbol{\Phi}}_n(u) := \boldsymbol{P}\left(\frac{u}{2}\right) \dots \boldsymbol{P}\left(\frac{u}{2^n}\right)\, \hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}\right)$ and with $\hat{\boldsymbol{\Phi}}_0$ defined as in (6.3).

Proof: 1. We have

$$\int_{-\infty}^{\infty} \|\hat{\boldsymbol{\Phi}}_n(u) - \hat{\boldsymbol{\Upsilon}}(u)\|\;\mathrm{d}u \leq \boldsymbol{S}_{1,n} + \boldsymbol{S}_{2,n} + \boldsymbol{S}_{3,n}$$

with

$$\begin{aligned}
\boldsymbol{S}_{1,n} &:= \int_{|u|\ge 2^n\pi} \|\hat{\boldsymbol{\Phi}}_n(u)\| \,\mathrm{d}u,\\
\boldsymbol{S}_{2,n} &:= \int_{|u|\ge 2^n\pi} \|\hat{\boldsymbol{\Upsilon}}(u)\| \,\mathrm{d}u,\\
\boldsymbol{S}_{3,n} &:= \int_{|u|\le 2^n\pi} \|\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\,\left[\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}\right)-\hat{\boldsymbol{\Upsilon}}\left(\frac{u}{2^n}\right)\right]\|\;\mathrm{d}u.
\end{aligned}$$

Since $\gamma_k < m-1$, it follows by Theorem 4.1 that $\|\hat{\boldsymbol{\Upsilon}}(u)\| \le C\,(1+|u|)^{-1-\epsilon}$ and hence $\lim_{n\to\infty}\boldsymbol{S}_{2,n}=0$.

2. Next, we discuss the convergence of $\boldsymbol{S}_{3,n}$. Using the relation (2.6) we obtain for any vector $\boldsymbol{v}$ of length r

$$\begin{aligned}
&\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\,\boldsymbol{v}\\
&= \frac{1}{2^n\,(1-e^{-iu/2^n})}\,\boldsymbol{C}\boldsymbol{x}_0(u)\,\boldsymbol{P}^{(1)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(1)}\left(\frac{u}{2^n}\right)\,\boldsymbol{E}\boldsymbol{x}_0\left(\frac{u}{2^n}\right)\,\boldsymbol{v}\\
&= \frac{(\boldsymbol{x}_0)^T\,\boldsymbol{v}}{2^n\,(1-e^{-iu/2^n})}\,\boldsymbol{C}\boldsymbol{x}_0(u)\,\boldsymbol{P}^{(1)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(1)}\left(\frac{u}{2^n}\right)\,\boldsymbol{e}_0\\
&\quad +\frac{-i}{2^n}\,\boldsymbol{C}\boldsymbol{x}_0(u)\,\boldsymbol{P}^{(1)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(1)}\left(\frac{u}{2^n}\right)\,(D\boldsymbol{E}\boldsymbol{x}_0)(0)\,\boldsymbol{v},
\end{aligned}$$

where $\boldsymbol{e}_0$ is defined by $\boldsymbol{x}_0$ via (4.2), and where we have used $(x_0)^T\boldsymbol{a}=1$. Consequently,

$$\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\,\left[\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}\right)-\hat{\boldsymbol{\Upsilon}}\left(\frac{u}{2^n}\right)\right]=\boldsymbol{A}(u)+\boldsymbol{B}(u)$$

with

$$\boldsymbol{A}(u) := (\boldsymbol{x}_0)^T\,\left[\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}\right)-\hat{\boldsymbol{\Upsilon}}\left(\frac{u}{2^n}\right)\right]\,\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\,\boldsymbol{C}\boldsymbol{x}_0\left(\frac{u}{2^n}\right)\,\boldsymbol{a}$$

and

$$\boldsymbol{B}(u) := \frac{1}{2^n}\,\boldsymbol{C}\boldsymbol{x}_0(u)\,\boldsymbol{P}^{(1)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(1)}\left(\frac{u}{2^n}\right)\,(\mathrm{D}\boldsymbol{E}\boldsymbol{x}_0)(0)\,\boldsymbol{\Xi}\left(\frac{u}{2^n}\right),$$

where $\boldsymbol{\Xi}(u)=\hat{\boldsymbol{\Phi}}(u)-\hat{\boldsymbol{\Upsilon}}(u)-(\boldsymbol{x}_0)^T\left[\hat{\boldsymbol{\Phi}}(u)-\hat{\boldsymbol{\Upsilon}}(u)\right]\boldsymbol{a}$. Now, we find by Hölder continuity of $\hat{\boldsymbol{\Phi}}(u)$ and $\hat{\boldsymbol{\Upsilon}}(u)$ at $u=0$,

$$\begin{aligned}
&\left|(\boldsymbol{x}_0)^T\,\left[\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}\right)-\hat{\boldsymbol{\Upsilon}}\left(\frac{u}{2^n}\right)\right]\right|\\
\le\;& \left|(\boldsymbol{x}_0)^T\,\left[\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}\right)-\hat{\boldsymbol{\Phi}}(0)\right]\right|+\left|(\boldsymbol{x}_0)^T\,\left[\hat{\boldsymbol{\Upsilon}}\left(\frac{u}{2^n}\right)-\hat{\boldsymbol{\Upsilon}}(0)\right]\right|\\
\le\;& C\,2^{-n}.
\end{aligned}$$

On the other hand, by the analysis we did before,

$$\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\|\le C\,(1+|u|)^{\gamma_k},$$

so that

$$\begin{aligned}
&\int_{|u|\le 2^n\pi} \left\|\boldsymbol{P}\left(\frac{u}{2}\right)\dots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}\right\|\,\mathrm{d}u \\
=\;& \int_{|u|\le 1} \left\|\boldsymbol{P}\left(\frac{u}{2}\right)\dots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}\right\|\,\mathrm{d}u + \int_{1\le|u|\le 2^n\pi} \left\|\boldsymbol{P}\left(\frac{u}{2}\right)\dots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}\right\|\,\mathrm{d}u \\
\le\;& C' + C'' \int_{1\le|u|\le 2^n\pi} \left|\frac{1}{2^n\,(1-e^{-iu/2^n})}\right|^m \left\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\right\|\,\mathrm{d}u \\
\le\;& C' + C''' \int_{1\le|u|\le 2^n\pi} (1+|u|)^{\gamma_k}\,|u|^{-m}\,\mathrm{d}u \\
\le\;& \tilde{C},
\end{aligned}$$

where we have used that $|1-e^{-iu/2^n}| = |\sin\frac{u}{2^{n+1}}|$ and $|\sin x| \ge \frac{2}{\pi}|x|$ for $|x| \le \frac{\pi}{2}$. Hence,

$$\int_{|u|\le 2^n\pi} \|\boldsymbol{A}(u)\|\,\mathrm{d}u \le \tilde{C}\,C\,2^{-n},$$

i.e, this term vanishes for $n\to\infty$.
For $\boldsymbol{B}(u)$, we have a similar estimate. We have $\|\boldsymbol{\Xi}\left(\frac{u}{2^n}\right)\| \le C2^{-n}$, and hence with the same arguments as above

$$\begin{aligned}
\int_{|u|\le 2^n\,\pi} \|\boldsymbol{B}(u)\|\,\mathrm{d}u \;&\le\; C'\,2^{-2n} \int_{|u|\le 2^n\pi} \left\|\boldsymbol{P}^{(1)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(1)}\left(\frac{u}{2^n}\right)\right\|\,\mathrm{d}u \\
&\le\; C'\,2^{-2n}\left(C'' + C''' \int_{1\le|u|\le 2^n\pi} |u|^{-m+1}\,(1+|u|)^{\gamma_k}\,\mathrm{d}u\right) \\
&\le\; C'\,2^{-2n}\,(C'' + \tilde{C}\,2^n\,\pi).
\end{aligned}$$

Thus, this term also vanishes for $n\to\infty$, so that

$$\lim_{n\to\infty}\boldsymbol{S}_{3,n} = 0.$$

3. Finally, we discuss $\boldsymbol{S}_{1,n}$. The $2^{n+1}\pi$ periodicity of $\boldsymbol{P}\left(\frac{u}{2}\right)\dots\boldsymbol{P}\left(\frac{u}{2^n}\right)$ implies that

$$\begin{aligned}
\boldsymbol{S}_{1,n} &= \int_{|u|\ge 2^n\pi} \left\|\boldsymbol{P}\left(\frac{u}{2}\right)\dots\boldsymbol{P}\left(\frac{u}{2^n}\right)\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}\right)\right\|\,\mathrm{d}u \\
&= \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_{|u|\le 2^n\pi} \left\|\boldsymbol{P}\left(\frac{u}{2}\right)\dots\boldsymbol{P}\left(\frac{u}{2^n}\right)\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}+2\pi k\right)\right\|\,\mathrm{d}u.
\end{aligned}$$

Now, using (6.1) and (6.2), we obtain for $|u|\le 2^n\pi$ and $k\ne 0$

$$\begin{aligned}
\left\|\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}+2\pi k\right)\right\| &= \left\|\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}+2\pi k\right) - \hat{\boldsymbol{\Phi}}(2\pi k)\right\| \\
&\le \frac{1}{2}\left|\frac{u}{2^n}\right|^2 \sup_{|t|\le\pi} \|(\mathrm{D}^2\hat{\boldsymbol{\Phi}})(2\pi k+t)\| \\
&\le C\left|\frac{u}{2^n}\right|^2 (1+|k|)^{-1-\epsilon}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\boldsymbol{S}_{1,n} &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_{|u|\leq 2^n\pi} \|\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\,\hat{\boldsymbol{\Phi}}\left(\frac{u}{2^n}+2\pi k\right)\|\;\mathrm{d}u \\
&\leq C\,2^{-2n}\int_{|u|\leq 2^n\pi} |u|^2\,\|\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\|\;\mathrm{d}u \\
&\leq C\,2^{-2n}\left(C_1 + C_2\int_{1\leq|u|\leq 2^n\pi} |u|^{-m+2}\,(1+|u|)^{\gamma_k}\;\mathrm{d}u\right) \\
&\leq C\,2^{-2n}\left(C_1+\tilde{C}\,(2^{n(-m+3+\gamma_k)}+1)\right) \\
&\leq C'\,2^{-2n}+C''\,2^{-n(m-1-\gamma_k)}.
\end{aligned}
$$

Since we have assumed that $\gamma_k < m-1$ it follows that $\boldsymbol{S}_{1,n}$ too vanishes for $n\to\infty$.
■

Remarks.
1. The same arguments work for L^2–convergence, except that we only need $\gamma_k < m-\frac{1}{2}$.
2. The graphs for the examples in section 7 are, in fact, graphs of close approximations $\boldsymbol{\Phi}_n$ to the true solutions $\boldsymbol{\phi}$, obtained by the iteration in Theorem 6.1.

7. Examples

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