Semigroup approach and maximal-regularity theory for the primitive equations

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Princeton–Tokyo Workshop
November 7, 2017
Primitive equations

governing equation describing large-scale motion of ocean and atmosphere

- box domain: $\Omega = (0, 1)^2 \times (-h, 0) =: G \times (-h, 0)$
- velocity: $u = (v, w) = (v_1, v_2, w)$
- horizontal derivative: $\nabla_H = (\partial_x, \partial_y)$, vertical derivative: $\partial_z$

3D Navier–Stokes equations

$$\begin{align*}
\frac{\partial}{\partial t} v + u \cdot \nabla v - \Delta v + \nabla_H \pi &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
\frac{\partial}{\partial t} w + u \cdot \nabla w - \Delta w + \partial_z \pi &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
\text{div} \ u &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
\left. u \right|_{t=0} &= a \quad \text{in} \quad \Omega.
\end{align*}$$

boundary conditions

$\Gamma_u, \Gamma_b, \Gamma_l$ mean upper, bottom, lateral boundaries, respectively.

$$\begin{align*}
\partial_z v = w = 0 \text{ on } \Gamma_u, \\
v = w = 0 \text{ on } \Gamma_b, \\
u: \text{ periodic on } \Gamma_l.
\end{align*}$$
governing equation describing large-scale motion of ocean and atmosphere

- box domain: \( \Omega = (0, 1)^2 \times (-h, 0) =: G \times (-h, 0) \)
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- horizontal derivative: \( \nabla_H = (\partial_x, \partial_y) \), vertical derivative: \( \partial_z \)

### 3D primitive equations (in simplified form)

\[
\frac{\partial v}{\partial t} + u \cdot \nabla v - \Delta v + \nabla_H \pi = 0 \quad \text{in} \quad \Omega \times (0, T),
\]

\[
\partial_z \pi = 0 \quad \text{in} \quad \Omega \times (0, T),
\]

\[
\text{div } u = 0 \quad \text{in} \quad \Omega \times (0, T),
\]

\[
u|_{t=0} = a \quad \text{in} \quad \Omega.
\]

### boundary conditions

\( \Gamma_u, \Gamma_b, \Gamma_l \) mean upper, bottom, lateral boundaries, respectively.

\[
\partial_z v = w = 0 \quad \text{on} \ \Gamma_u, \quad v = w = 0 \quad \text{on} \ \Gamma_b, \quad u: \text{periodic on} \ \Gamma_l.
\]
Anisotropic nonlinearity

We remove $w$ (diagnostic variable) by

$$\text{div}_H \nu + \partial_z w = 0, \quad w|_{z=0} = 0 \quad \Rightarrow \quad w = \int_z^0 \text{div}_H \nu \, d\zeta.$$  

Recalling that $u \cdot \nabla = \nu \cdot \nabla_H + w \partial_z$, we have:

reformulated primitive equations

$$\partial_t \nu + \nu \cdot \nabla_H \nu + (\int_z^0 \text{div}_H \nu \, d\zeta) \partial_z \nu - \Delta \nu + \nabla_H \pi = 0$$

The nonlinearity is of quadratic-gradient type: "worse" than the usual Navier–Stokes equations (but not worse for the $z$-direction)

constraint

Since $w|_{z=-h} = 0$, $\int_{-h}^{0} \text{div}_H \nu \, dz = 0$. If $\overline{\nu}$ denotes the vertical average, 

$$\text{div}_H \overline{\nu} = 0 \quad \text{in} \quad G.$$  

boundary conditions (Neumann–Dirichlet type)

$$\partial_z \nu = 0 \text{ on } \Gamma_u, \quad \nu = 0 \text{ on } \Gamma_b, \quad \nu \text{ is periodic on } \Gamma_I.$$
Known results

  mathematical formulation, existence of a weak solution

- Guillén-González, Masmoudi and Rodríguez-Bellido (2001):
  local (in time) well-posedness of a strong solution for \( a \in H^1 \)

- Cao and Titi (2007):
  global well-posedness of the strong solution \((H^1-a\text{ priori estimate})\)

- Kucavica and Ziane (2007):
  Neumann–Dirichlet boundary conditions

- Kukavica, Pei, Rusin and Ziane (2014), Li and Titi (2015):
  when \( a \in C(\Omega) \)
Our approach (2014–)

What about if we employ $L^p (p \neq 2)$-based spaces?

→ analytic semigroup approach

- The same question for the Navier–Stokes was addressed.
- If $a \in L^3_\sigma(\Omega)$, then there exists a unique local strong solution.

Question

- Can we obtain analogous results for the primitive equations?
- Is it possible to obtain a strong solution for $a$ with no differentiability?

Our results

1. Fujita–Kato method works for $a \in H^{2/p,p}$:

2. Also works for $L^\infty_{xy}L^p_z$-type spaces (in progress):
   - Neumann–Dirichlet case: $p > 3$
   - Neumann–Neumann case: $p = 1
Global strong well-posedness for $H^{2/p, p}$-initial data for $1 < p < \infty$

(Hieber–K. ARMA 2016)
Resolvent problem (linear stationary problem)

\[ \Sigma_\theta := \{ \lambda \in \mathbb{C} : |\arg \lambda| < \theta \}. \]

**Theorem**

For \( \lambda \in \Sigma_{\pi - \epsilon} \cup \{0\} \), \( 1 < p < \infty \) and \( f \in L^p(\Omega)^2 \), there exists a unique solution \((v, \pi) \in W^{2,p}(\Omega)^2 \times W^{1,p}(G)\) of

\[
\begin{align*}
\lambda v - \Delta v + \nabla_H \pi &= f \quad \text{in} \quad \Omega, \\
\text{div}_H \bar{v} &= 0 \quad \text{in} \quad G, \\
\partial_z v &= 0 \quad \text{on} \quad \Gamma_u, \\
v &= 0 \quad \text{on} \quad \Gamma_b,
\end{align*}
\]

\( v \) and \( \pi \) are periodic on \( \Gamma_l \).

Moreover,

\[
|\lambda| \|v\|_{L^p(\Omega)} + \|v\|_{W^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(G)} \leq C \|f\|_{L^p(\Omega)}. 
\]

**Idea of proof:**

Step 1: Prove case \( p = 2 \) by the Lax–Milgram theorem.

Step 2: General \( p \) can be treated by a bootstrapping argument combined with the \( L^p \)-resolvent estimate for the usual 2D Stokes equations.
hydrostatic Helmholtz projector and Stokes operator

\[ W_{\text{per}}^{1,p}(G) := \{ \nu \in W^{1,p}(G) \mid \nu \text{ is periodic on } \partial G \} \].

**Proposition**

For \( f \in L^p(G)^2 \), there exists a unique weak solution \( \pi \in W_{\text{per}}^{1,p}(G) \) of

\[ \Delta_H \pi = \text{div}_H f \quad \text{in} \quad G. \]

**Definition**

(1) Define a bounded linear operator \( P \) on \( L^p(\Omega)^2 \) by

\[ Pf := f - \nabla_H \pi. \]

(2) \( L^p_\sigma(\Omega) := \text{Ran } P = \{ \nu \in L^p(\Omega)^2 : \text{div}_H \bar{\nu} = 0, \quad \bar{\nu} \cdot \nu_{\partial G} \text{ is periodic} \} \)

**Theorem**

Define a linear operator \( A \) on \( L^p_\sigma(\Omega) \) by \( Av := -P \Delta \nu \), together with

\[ D(A) := \{ \nu \in W_{\text{per}}^{2,p}(\Omega)^2 : \text{div}_H \bar{\nu} = 0 \text{ in } G, \quad \partial_z \nu = 0 \text{ on } \Gamma_u, \quad \nu = 0 \text{ on } \Gamma_b \}. \]

Then \( A \) generates a \( C^0 \)-analytic semigroup \( e^{-tA} \).
Consequences of analyticity

For $\theta \in [0, 1]$ we set $V_\theta := [X, D(A)]_\theta \subset H^{2\theta,p}(\Omega)^2(\approx W^{2\theta,p}(\Omega)^2)$.

**Characterization (Hieber–Hussein–K., 2017)**

- $0 \leq \theta < 1/2p$:
  $$V_\theta = \{ \nu \in H^{2\theta,p}_{\text{per}}(\Omega)^2 \mid \text{div}_H \nu = 0 \}$$

- $1/2p < \theta < 1/2 + 1/2p$:
  $$V_\theta = \{ \nu \in H^{2\theta,p}_{\text{per}}(\Omega)^2 \mid \text{div}_H \nu = 0, \quad \nu|_{\Gamma_b} = 0 \}$$

- $1/2 + 1/2p < \theta \leq 1$:
  $$V_\theta = \{ \nu \in H^{2\theta,p}_{\text{per}}(\Omega)^2 \mid \text{div}_H \nu = 0, \quad \nu|_{\Gamma_b} = 0, \quad \partial_z \nu|_{\Gamma_u} = 0 \}$$

Example: when $p = 2$, $V_{1/2} = \{ \nu \in H^1(\Omega)^2 \mid \text{div}_H \nu = 0, \quad \nu|_{\Gamma_b} = 0 \}$.

**Corollary of analyticity of $A$**

$$\|e^{-tA}a\|_{V_{\theta_1+\theta_2}} \leq Ct^{-\theta_1}e^{-\frac{3}{2}\beta t}\|a\|_{V_{\theta_2}}$$

temporal divergence factor $\leftrightarrow 2 \times$ spatial differentiability
Local strong solution

**Theorem**

Let \( a \in V_{1/p} \). Then, for some \( T^* > 0 \) there exists a unique solution of the primitive equations for \( t \in (0, T^*] \) such that

\[
\nu \in C([0, T^*]; V_{1/p}) \cap C^1((0, T^*]; X) \cap C((0, T^*]; D(A))
\]

When \( p = 2 \), \( V_{1/2} \) agrees with the space known in the previous works.

**Lower bound of \( T^* \)**

\[
T^* \geq \left( \frac{1}{C \|a\|_{V_{1/p+\epsilon,p}}} \right)^{1/\epsilon}, \quad 0 < \epsilon \leq 1 - 1/p.
\]

However, \( T^* \) cannot be controlled solely by \( \|a\|_{V_{1/p,p}} \).

Later we use this with \( p = 2 \) and \( \epsilon = 1/2 \).

**Regularity required for initial data**

- \( p = 2: a \in H^1 \)
- \( p = 3: a \in H^{2/3,3} \)
- \( p = \infty: a \in L^\infty \) (formally)
Remark on local existence theorem by Galerkin’s method

The class of the strong solution is:

\[ \nu \in C([0; T^*]; H^1(\Omega)) \cap L^2(0; T^*; H^2(\Omega)). \]

Navier–Stokes case

Since \( \frac{1}{2} \frac{d}{dt} \| \nabla \nu \|_{L^2}^2 + \| \Delta \nu \|_{L^2}^2 \leq C \| \nabla \nu \|_{L^2}^6 \), we have

\[
\sup_{0 \leq t \leq T^*} \| \nu(t) \|_{H^1}^2 + 2 \int_0^{T^*} \| \nu \|_{H^2}^2 \, dt \leq \| a \|_{H^1}^2 + C T^* \left( \sup_{0 \leq t \leq T^*} \| \nu(t) \|_{H^1} \right)^6.
\]

Therefore, if the maximum existence time \( T_{\max} \) is finite, then

\[
\lim_{t \to T_{\max}} \| \nu(t) \|_{H^1} = \infty.
\]

However, in the primitive equations, the possibility that

\[
\lim_{t \to T_{\max}} \| \nu(t) \|_{H^1} < \infty \quad \text{but} \quad \lim_{t \to T_{\max}} \int_0^{T^*} \| \nu \|_{H^2}^2 \, dt = \infty
\]

may not be excluded by the local existence theorem.
Idea of proof

\[ \partial_t v + v \cdot \nabla_H v + w \partial_z v - \Delta v + \nabla_H \pi = 0 \]

\[ \iff \partial_t v + Av = F(v) := -P(v \cdot \nabla_H v + w \partial_z v) \]

\[ \iff v(t) = e^{-tA}a + \int_0^t e^{-A(t-s)}F(v(s)) \, ds. \]

It suffices to construct \( v \) satisfying this integral equation (mild solution).

**Iteration scheme**

(i) \( v_0(t) = e^{-tA}a \)

(ii) \( v_m(t) = e^{-tA}a + \int_0^t e^{-A(t-s)}F(v_{m-1}(s)) \, ds, \quad m = 1, 2, \ldots \)

**Goal:** prove that \( \{v_m\} \) forms a Cauchy sequence in some Banach space.

**Proposition**

\[ \|F(v)\|_{L^p(\Omega)} \leq C\|v\|_{V_{1/2+1/2p}}^2. \]

**Example (\( p = 2 \))**:

\[ \|w \partial_z v\|_{L^2} \leq C\|w\|_{L^4_{xy}L^\infty_z} \|\partial_z v\|_{L^4_{xy}L^2_z} \leq C\|\partial_z w\|_{H_{xy}^{1/2}L^2_z} \|\partial_z v\|_{H_{xy}^{1/2}L^2_z} \leq C\|v\|_{H^{3/2}}^2. \]
Idea of proof (cont.)

Then we obtain for $0 < t \leq T$

$$t^{1/2-1/2p} \|v_m(t)\| v_{1/2+1/2p} \leq C \|a\| v_{1/p} + C \left( \sup_{0 \leq s \leq t} s^{1/2-1/2p} \|v_{m-1}(s)\| v_{1/2+1/2p} \right)^2$$

Assume one of the following:

(i) $\|a\| v_{1/p}$ is small
(ii) $T$ is small

Then it follows that

$$\sup_{0 \leq s \leq T} s^{1/2-1/2p} \|v_{m-1}(s)\| \leq \exists M \text{ and } M \text{ is sufficiently small.}$$

This implies that $\{v_m\}$ is Cauchy.
Global strong solution: a priori $H^2$-estimate

Aim

- Derive a priori bound for $v \in L^\infty(0, T; H^2(\Omega)^2)$.
- We need care to treat the Dirichlet boundary condition on $\Gamma_b$.

Step 0: estimate for $v \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$

Step 1: estimate for $\tilde{v} \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$, $\nabla H \pi \in L^2(0, T; L^2)$

Step 2: estimate for $\partial_z v \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$

Step 3: estimate for $\tilde{v} := v - \tilde{v} \in L^\infty(0, T; L^4)$

Step 4: Each of Steps 1–3 is not closed, but adding them enables us to absorb all the bad terms.

Compare

- Cao and Titi (2007): $\tilde{v} \in L^\infty(L^6)$ in Step 3
- Kukavica and Ziane (2007): $\nabla H \pi \in L^2(L^{3/2})$ in Step 1
Step 1:

$$8 \partial_t \| \nabla_H \tilde{v} \|_{L^2}^2 + \| \nabla H \pi \|_{L^2}^2 \leq C_1 \| \tilde{v} \|_{L^2} \| \nabla \tilde{v} \|_{L^2}^2 + \frac{1}{4} \| \nabla \partial_z v \|_{L^2}^2 + \text{(others)}$$

Step 2:

$$\partial_t \| \partial_z v \|_{L^2}^2 + \| \nabla \partial_z v \|_{L^2}^2 \leq C \| v \|_{H^1}^2 \| \partial_z v \|_{L^2}^2 + \frac{1}{2} \| \nabla H \pi \|_{L^2}^2 + C_2 \| \tilde{v} \|_{L^2} \| \nabla \tilde{v} \|_{L^2}^2 + \text{(others)}$$

Step 3:

$$\frac{C_1 + C_2}{4} \partial_t \| \tilde{v} \|_{L^4}^4 + (C_1 + C_2) \| \tilde{v} \|_{L^2} \| \nabla \tilde{v} \|_{L^2}^2 \leq C \| v \|_{H^1}^2 \| \tilde{v} \|_{L^4}^4 + \frac{1}{4} \| \nabla \partial_z v \|_{L^2}^2 + \text{(others)}$$

Neumann-Neumann case

$$\frac{1}{4} \| \nabla \partial_z v \|_{L^2}^2$$ in Steps 1 and 3 is absent.
Global strong solution (cont.)

Step 5: estimate for \( v \in L^{\infty}(0, T; H^1) \cap L^2(0, T; H^2) \)

Remark: strong solution based on Galerkin's method

Step 5 is the final result.

Step 6: estimate for \( \partial_t v \in L^{\infty}(0, T; L^2) \cap L^2(0, T; H^1) \)

Step 7: estimate for \( v \in L^{\infty}(0, T; H^2) \)

Step 8: global existence for \( p = 2 \):

- We may regard \( v(\{3t_1\}) \in H^2 \) as new initial data.
- By the a priori estimate, \( \sup_{0 \leq t \leq T^*} \| v(t) \|_{H^2} \leq 3B \).
- Local existence theorem extends the solution from \( T^* \) to \( T^* + TB \).
- the above argument can be repeated to reach any \( T > 0 \).

Step 9: Global existence for another \( p \) follows from bootstrap arguments:

\[
\partial_t v - \Delta v + \nabla_H \pi = v \cdot \nabla_H v + w \partial_z v.
\]
## Theorem (Hieber–K. 2016)

Let $p \in (1, \infty)$ and $a \in V_{1/p} \subset H^{2/p,p}(\Omega)^2$. Then there exists a unique strong solution to the primitive equations such that

$$\nu \in C([0, \infty); V_{1/p}) \cap C^1((0, \infty); L^p_0(\Omega)) \cap C((0, \infty); D(A))$$

Moreover, $\|\nu(t)\|_{D(A)} \leq Ce^{-ct}$.

## Extensions

- **maximal regularity (with time weights)**: Hussein (in progress)
- **$C^\infty$ or real analyticity in $(x, t)$**: Hussein et al. (in progress)
- **time-periodic sol. with large force**: Galdi–Hieber–K. (submitted)
Recent developments: global strong well-posedness for $L^\infty$-type initial data
Remark on initial data with no differentiability

- Recalling that the nonlinearity of the primitive equation is of \textit{quadratic gradient} type, we consider
  \[ \partial_t u - \Delta u = |\nabla u|^2, \quad u(0) = u_0 \quad \text{in} \quad \mathbb{R}. \]
- This has the following explicit solution (Evans):
  \[ u(x, t) = \log \left( \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} + u_0(y) \, dy \right). \]
- Suppose this is well-posed for \( u_0 \in L^p \) with \( p < \infty \); then \( u_0(y) = |y|^{-\alpha} (\exists \alpha > 0) \) must be allowed.
- \( \int e^{|y|^{-\alpha}} \, dy \) diverges because \( |y|^{-1} \leq e^{|y|^{-\alpha}} \) holds near \( y = 0 \).
- However, if \( u_0 \in L^\infty \), the integral is finite.

Conjecture

- \( L^p \) (\( p < \infty \)) may be hopeless.
- It may be possible to construct strong solution with \( a \in L^\infty(\Omega)^2 \).
- \( H^{2/p, p} \) formally becomes \( L^\infty \) as \( p \to \infty \).
Neumann–Neumann case

- $A$ is essentially Laplacian, and $e^{-tA}$ is essentially heat semigroup.
- Heat semigroup in $L^\infty$-type space can be explicitly treated.
- Helmholtz projector $P$ is not bounded in $L^\infty(G)^2$.
- Fractional powers of $\Delta_H$ and $\partial_z$ are utilized to estimate $F(v)$.

**Lemma**

$$\|e^{-tA}F(v)\|_{L^\infty_{xy}L^1_z} \leq Ct^{-1/4}\|\nabla v\|_{L^\infty_{xy}L^1_z}^{3/2}\|v\|_{L^\infty_{xy}L^1_z}^{1/2}.$$ 

- With this we obtain quadratic inequalities for $\sup_{0 \leq t \leq T} t^{1/2}\|\nabla v_m(t)\|_{L^\infty_{xy}L^1_z}$ and $\sup_{0 \leq t \leq T} \|v_m(t)\|_{L^\infty_{xy}L^1_z}$.
- Assuming $\|a\|_{L^\infty_{xy}L^1_z}$ or $T$ is small, we have uniform bounds.

**Theorem (in progress)**

For $a \in C_{xy}L^1_z$, there exists a unique solution of the primitive equations such that $C([0, \infty); C_{xy}L^1_z) \cap C^\infty(\overline{\Omega} \times (0, \infty))$.

cf. Li–Titi (2015): $a \in L^6_{xy}H^1_z \cup C(\overline{\Omega})$. 
Neumann–Dirichlet case

- $A$ is not Laplacian.
- $e^{-tA}$ in $L^\infty$-type spaces has to be studied indirectly.

**Lemma (in progress)**

(i) $e^{-tA}$ is a bounded analytic semigroup in $L_{xy}^\infty L_z^p \cap L_{\partial}^p$ for $3 < p < \infty$.
(ii) $\|\partial_i e^{-tA}Pf\|_{L_{xy}^\infty L_z^p} \leq C t^{-1/2} \|f\|_{L_{xy}^\infty L_z^p}$ for $i = x, y$.
(iii) $\|\partial_z e^{-tA}Pf\|_{L_{xy}^\infty L_z^p} \leq C t^{-1/2} \|f\|_{L_{xy}^\infty L_z^p}$.
(iv) $\|e^{-tA}P\partial_z f\|_{L_{xy}^\infty L_z^p} \leq C t^{-1/2} \|f\|_{L_{xy}^\infty L_z^p}$ for $f|_{\Gamma_u \cup \Gamma_b} = 0$.

- For $a \in C_{xy} L_z^p$, we decompose $a = a_{\text{smooth, big}} + a_{\text{rough, small}}$.
- We consider a perturbed equation corresponding to $a_{\text{rough, small}}$ (cf. Kukavica et al. 2014, Li–Titi 2015).
- Construct an iteration scheme for the perturbed equation.
- Uniform bounds can be derived because $a_{\text{rough, small}}$ is small.
We constructed a strong solution of the primitive equations using the semigroup approach.

The approach has an advantage in determining an admissible class of initial data.

\[ L^p \text{ setting: } a \in H^{2/p}p(\Omega)^2 \]

\[ L^\infty \text{-type setting:} \]

- Neumann–Neumann case: \( a \in C_{xy}L_z^1 \)
- Neumann–Dirichlet case: \( a \in C_{xy}L_z^p \) (\( 3 < p < \infty \))

### Contribution to application side

- Is rough initial data useful?
- Coriolis force
- Spherical domain ... etc