On the Navier–Stokes equations with Navier boundary conditions in a curved thin domain

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Thin domain

Thin domain $\Omega_\varepsilon$ in $\mathbb{R}^n$ ($n \geq 2$, $\varepsilon > 0$: small)
  - domain with very small width of order $\varepsilon$ in some directions
  - degenerating into a lower dimensional set as $\varepsilon \to 0$

Fluid flows in $\Omega_\varepsilon$ appears in many problems of natural sciences
  - ocean dynamics, geophysical fluid dynamics
  - dynamics of fluid flows in cell membranes
Navier–Stokes equations in a 3D thin domain

Topics of the Navier–Stokes equations in a 3D thin domain $\Omega_\varepsilon$

- global-in-time existence of strong solutions for large data
- derivation and analysis of limit equations as $\varepsilon \to 0$
- comparison of long time behavior of solutions to the Navier–Stokes equations and the limit equations

Mathematical studies of the 3D Navier–Stokes equations in $\Omega_\varepsilon$

  - a flat thin domain, e.g. $\Omega_\varepsilon = (0, 1)^2 \times (0, \varepsilon)$
  - a thin spherical shell $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid 1 < |x| < 1 + \varepsilon\}$

T.-H. Miura (Univ. of Tokyo)  NS eqs. in a curved thin domain  2017/11/8  3 / 15
Curved moving thin domain $\Omega_\varepsilon(t)$ in $\mathbb{R}^3$ ($t \in [0, T]$)

- $\Omega_\varepsilon(t) \to \Gamma(t)$ ($\varepsilon \to 0$): given closed evolving 2D surface
- $\Gamma(t)$ is connected, oriented and has no singularity

In this talk, we only consider an $\varepsilon$-tubular neighborhood of $\Gamma(t)$

$$\Omega_\varepsilon(t) = \{ x \in \mathbb{R}^n \mid \text{dist}(x, \Gamma(t)) < \varepsilon \}$$
Mathematical analysis of PDEs in a curved moving thin domain $\Omega_\varepsilon(t)$

  - diffuse interface model for an advection-diffusion equation
  - the advection-diffusion equation on $\Gamma(t)$ is given

- M. (2017, Interfaces Free Bound.)
  - Neumann type problem of the heat equation in $\Omega_\varepsilon(t)$
  - the limit equation on $\Gamma(t)$ is derived

A formal derivation of limit equations for nonlinear PDEs in $\Omega_\varepsilon(t)$

- M.–Giga–Liu (Hokkaido Univ. Preprint Series in Math. # 1101)
  - a nonlinear diffusion equation of porous media type

  - the incompressible Euler and Navier–Stokes equations
Evolving surface and curved thin domain

- $\Gamma(t), \ t \in [0, T]$: given closed smooth evolving 2D surface in $\mathbb{R}^3$
  - $\nu(\cdot, t)$: unit outward normal vector to $\Gamma(t)$
  - $V_N^\Gamma(\cdot, t)$: outward normal velocity of $\Gamma(t)$

- $\Omega_\varepsilon(t), \ t \in [0, T]$: curved moving thin domain in $\mathbb{R}^3$
  $$\Omega_\varepsilon(t) = \{ x \in \mathbb{R}^3 \mid \text{dist}(x, \Gamma(t)) < \varepsilon \} \quad (\varepsilon > 0)$$
  - $\nu_\varepsilon(\cdot, t)$: unit outward normal vector to $\partial \Omega_\varepsilon(t)$
  - $V_N^\varepsilon(\cdot, t)$: outward normal velocity of $\partial \Omega_\varepsilon(t)$

$$ (\nu_\varepsilon, V_N^\varepsilon)(x, t) = \pm (\nu, V_N^\Gamma)(\pi(x, t), t), \quad d(x, t) = \pm \varepsilon $$

$$ \begin{pmatrix} \pi(\cdot, t): \text{closest point mapping onto } \Gamma(t) \\ d(\cdot, t): \text{signed distance function from } \Gamma(t) \end{pmatrix} $$
The 3D Navier–Stokes equations with Navier boundary conditions

\[
\begin{aligned}
&\partial_t u + (u \cdot \nabla)u + \nabla p = \mu_0 \Delta u \quad \text{in} \quad \Omega_\varepsilon(t), \; t \in (0, T), \\
&\text{div} \; u = 0 \quad \text{in} \quad \Omega_\varepsilon(t), \; t \in (0, T), \\
&u \cdot \nu_\varepsilon = V_\varepsilon^N \quad \text{on} \quad \partial\Omega_\varepsilon(t), \; t \in (0, T), \\
&[D(u)\nu_\varepsilon]_{\text{tan}} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon(t), \; t \in (0, T).
\end{aligned}
\]

- $\mu_0 > 0$: viscosity coefficient
- $D(u) = \{\nabla u + (\nabla u)^T\}/2$: strain rate tensor
- $[a]_{\text{tan}} = (I_3 - \nu_\varepsilon \otimes \nu_\varepsilon)a$ on $\partial\Omega_\varepsilon(t)$ for $a \in \mathbb{R}^3$
We would like to derive limit equations of \((N S_\varepsilon)\), i.e. equations on \(\Gamma(t)\) satisfied by the thin width limit of the average

\[
M_\varepsilon u(y, t) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u(y + r\nu(y, t), t)dr, \quad y \in \Gamma(t).
\]

We assume that \(\Gamma(t)\) admits the normal coordinate system

\[
\begin{align*}
x &= \pi(x, t) + d(x, t)\nu(\pi(x, t), t), \quad x \in \Omega_\varepsilon(t) \\
\begin{pmatrix}
\pi(\cdot, t) : \text{closest point mapping onto } \Gamma(t) \\
d(\cdot, t) : \text{signed distance function from } \Gamma(t)
\end{pmatrix}
\end{align*}
\]

and consider the Taylor series of

\[
u(x, t) = u(\pi(x, t) + d(x, t)\nu(\pi(x, t), t), t)
\]

and \(p(x, t)\) with respect to the signed distance \(d(x, t)\).
In the Taylor series of $u$ and $p$

$$u(x, t) = v(\pi, t) + d(x, t)v^1(\pi, t) + d(x, t)^2v^2(\pi, t) + \cdots,$$

$$p(x, t) = q(\pi, t) + d(x, t)q^1(\pi, t) + d(x, t)^2q^2(\pi, t) + \cdots$$

($v(\pi, t) = v(\pi(x, t), t)$ for $x \in \Omega_\varepsilon(t)$, etc.)

we assume that the coefficients are independent of $\varepsilon$. Then

$$M_\varepsilon u(y, t) = v(y, t) + \text{(higher order terms in } \varepsilon), \quad y \in \Gamma(t).$$

Thus, formally speaking, the limit equations of $(NS_\varepsilon)$ are given as equations on $\Gamma(t)$ satisfied by $v(y, t)$ and $q(y, t)$. 
To derive the limit equations, we
\(\triangleright\) substitute the series of \(u(x, t)\) and \(p(x, t)\) for \((NS_\epsilon)\),
\(\triangleright\) use the differentiation formulas
\[
\nabla(f(\pi(x, t), t)) = \nabla_{\Gamma} f(\pi, t) + d(x, t)[A \nabla_{\Gamma} f](\pi, t) + \cdots,
\]
\[
\partial_t(f(\pi(x, t), t)) = \partial^o f(\pi, t)
\]
\[
+ d(x, t)[(\nabla_{\Gamma} V_{\Gamma}^N \cdot \nabla_{\Gamma}) f](\pi, t) + \cdots
\]
\[
\begin{pmatrix}
\nabla_{\Gamma} f = (I_3 - \nu \otimes \nu) \nabla f
\ \\
A = -\nabla_{\Gamma} \nu
\ \\
\partial^o f = \partial_t f + V_{\Gamma}^N \nu \cdot \nabla f
\end{pmatrix}
\]
\(\triangleright\) for \(f(\cdot, t) : \Gamma(t) \rightarrow \mathbb{R}\) and \(x \in \Omega_\epsilon(t) (\pi = \pi(x, t))\), and
determine the zeroth order term of \((NS_\epsilon)\) in \(d(x, t)\).
The limit equations of the Navier–Stokes equations

\[
\begin{cases}
\partial_v \nu + \nabla_\Gamma q + q^1 \nu = 2\mu_0 \text{div}_\Gamma[D_\Gamma(v)], \\
\text{div}_\Gamma v = 0, \quad v \cdot \nu = V^N_\Gamma \quad \text{on} \quad \Gamma(t), \ t \in (0, T)
\end{cases}
\]

for \(v(\cdot, t): \Gamma(t) \to \mathbb{R}^3\) and \(q(\cdot, t), q^1(\cdot, t): \Gamma(t) \to \mathbb{R}\).

\[\partial_v \nu = \partial^\circ \nu + (\nu \cdot \nabla_\Gamma) v: \text{material derivative}\]

\[\text{div}_\Gamma v = \text{tr} [\nabla_\Gamma v]: \text{surface divergence}\]

\[P_\Gamma: \text{orthogonal projection onto the tangent plane of } \Gamma(t)\]

\[D_\Gamma(v): \text{surface strain rate tensor}\]

\[D_\Gamma(v) = \frac{1}{2} P_\Gamma \{\nabla_\Gamma v + (\nabla_\Gamma v)^T\} P_\Gamma\]
Remarks on the limit equations

\[(NS_0)\] \[
\begin{align*}
\partial_v v + \nabla_\Gamma q + q^1 \nu &= 2\mu_0 \text{div}_\Gamma [D_\Gamma (v)], \\
\text{div}_\Gamma v &= 0, \quad v \cdot \nu = V^N_\Gamma \quad \text{on} \quad \Gamma(t), \; t \in (0, T).
\end{align*}
\]

- The normal component of \(v\) is given by \(v \cdot \nu = V^N_\Gamma\), i.e. the normal velocity of a surface fluid is the same as that of \(\Gamma(t)\).

- The tangential component of \(v\) and \(q\) are determined by

\[(TNS_0)\] \[
\begin{align*}
P_\Gamma \partial_v v + \nabla_\Gamma q &= 2\mu_0 P_\Gamma \text{div}_\Gamma [D_\Gamma (v)], \\
\text{div}_\Gamma v &= 0 \quad \text{on} \quad \Gamma(t), \; t \in (0, T).
\end{align*}
\]

- Finally, \(q^1\) is given by \(q^1 = \{ -\partial_v v + 2\mu_0 \text{div}_\Gamma [D_\Gamma (v)] \} \cdot \nu\).
The equations for incompressible fluids on an evolving surface

\[ (NS_0)' \begin{cases} \partial_v \cdot v + \nabla_\Gamma q + qHv = 2\mu_0 \text{div}_\Gamma [D_\Gamma (v)], \\ \text{div}_\Gamma v = 0, \quad v \cdot v = V_\Gamma^N \quad \text{on} \quad \Gamma(t), \quad t \in (0, T) \end{cases} \]

\( H(\cdot, t) \): mean curvature of \( \Gamma(t) \) is derived by


The momentum equation is of the form \( \partial_v \cdot v = \text{div}_\Gamma S_\Gamma \), where

\[ S_\Gamma = -qP_\Gamma + (\lambda_0 - \mu_0)(\text{div}_\Gamma v)P_\Gamma + 2\mu_0 D_\Gamma (v) \]

is the Boussinesq–Scriven surface stress tensor.

It appears in the study of two-phase flows, e.g.

Comparison with the NS eqs. on a stationary manifold

\[
\begin{align*}
(TNS_0) \left\{ \begin{array}{l}
P_{\Gamma}\partial_t v + \nabla_{\Gamma} q = 2\mu_0 P_{\Gamma}\text{div}_{\Gamma}[D_{\Gamma}(v)], \\
\text{div}_{\Gamma} v = 0 \quad \text{on} \quad \Gamma(t), \ t \in (0, T).
\end{array} \right.
\end{align*}
\]

When \( \Gamma(t) = \Gamma \) is stationary, \( v \cdot \nu = 0 \) and \((TNS_0)\) becomes

\[
\begin{align*}
(TNS_0)' \left\{ \begin{array}{l}
\partial_t v + \nabla_{\nu} v + \nabla_{\Gamma} q = \mu_0 (\Delta_B v + K v), \\
\text{div}_{\Gamma} v = 0 \quad \text{on} \quad \Gamma \times (0, T).
\end{array} \right.
\end{align*}
\]

\(\nabla: \text{Levi–Civita connection}, \ K: \text{Gaussian curvature}\)

\(\Delta_B = -\nabla^* \circ \nabla: \text{Bochner Laplacian}\)

\(\nabla: \text{Levi–Civita connection}, \ K: \text{Gaussian curvature}\)

\[
\begin{align*}
(TNS_0)' \text{ is the same as the Navier–Stokes eqs. on a manifold given by Taylor (1992, Comm. Partial Differential Equations).}
\end{align*}
\]
Lemma
For a tangential vector field $\nu$ on $\Gamma$ satisfying $\text{div}_\Gamma \nu = 0$,

$$2P_\Gamma \text{div}_\Gamma [D_\Gamma (\nu)] = P_\Gamma (\Delta_\Gamma \nu) + HA\nu \quad \text{on} \quad \Gamma.$$ 

Lemma
For tangential vector fields $X$ and $Y$ on $\Gamma$,

$$\begin{cases} \quad \nabla_Y X = (Y \cdot \nabla)X - (AX \cdot Y)\nu = P_\Gamma \{(Y \cdot \nabla)X\} \\
\Delta_B X = P_\Gamma (\Delta_\Gamma X) + A^2 X \end{cases} \quad \text{on} \quad \Gamma.$$ 

- $\Delta_\Gamma = \text{div}_\Gamma \nabla_\Gamma$: Laplace–Beltrami operator on $\Gamma$
- $A = -\nabla_\Gamma \nu$: Weingarten map (shape operator) of $\Gamma$
- $HA\nu = Kv + A^2 \nu$ for every tangential vector field $\nu$ on $\Gamma$