Global strong solutions to the equations governing the motion of nonhomogeneous fluids

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The nonhomogeneous incompressible MHD equations reads

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}(2\mu(\rho)d) + \nabla P - (\nabla \times B) \times B &= 0, \\
\partial_t B - \nabla \times (u \times B) + \nabla \times (\lambda(\rho)\nabla \times B) &= 0, \quad \text{(MHD)} \\
\text{div}u &= 0, \quad \text{div}B = 0.
\end{align*}
\]

- \(\rho\): density, \(u\): velocity field, \(B\): magnetic field, \(P\): pressure,
- \(d := \frac{1}{2}[\nabla u + (\nabla u)^T]\): deformation tensor, \(\mu(\rho)\) and \(\lambda(\rho)\) stand for the viscosity and resistivity coefficients respectively, and are both functions of density \(\rho\) satisfying

\[
\mu \in C^1[0, \infty), \quad \mu \geq \underline{\mu} > 0, \quad \text{and} \quad \lambda \in C^1[0, \infty), \quad \lambda \geq \underline{\lambda} > 0. \quad (1)
\]
In this talk, we focus on the nonhomogeneous MHD over a bounded domain \( \Omega \) (unit normal vector \( \vec{n} \)) with following initial boundary conditions:

\[
\begin{align*}
    u &= 0, & B \cdot \vec{n} &= 0, & (\nabla \times B) \times \vec{n} &= 0, & \text{on } \partial \Omega \times [0, T), \\
    (\rho, u, B)|_{t=0} &= (\rho_0, u_0, B_0) & \text{in } \Omega.
\end{align*}
\]
In this talk, we focus on the nonhomogeneous MHD over a bounded domain $\Omega$ (unit normal vector $\vec{n}$) with following initial boundary conditions:

$$u = 0, \quad B \cdot \vec{n} = 0, \quad (\nabla \times B) \times \vec{n} = 0, \quad \text{on } \partial \Omega \times [0, T),$$

$$(\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0) \quad \text{in } \Omega. \quad (3)$$

**Remark**

Under the assumption that $\partial \Omega$ is smooth, we have the following fact which is a consequence of boundary condition $B \cdot n|_{\partial \Omega} = 0$ and divergence free property of magnetic field $B$.

$$\int_{\Omega} B dx = \int_{\Omega} \text{div}(x \otimes B) dx = \int_{\partial \Omega} x (B \cdot n) dS = 0.$$

Here $x \otimes B$ is a matrix with $i, j$ component $x_i B_j$. Therefore we can also apply the Poincaré inequality for magnetic field $B$. 
The nonhomogeneous incompressible MHD system is a combination of the nonhomogeneous Navier-Stokes equations of fluid mechanics and Maxwell equations of electromagnetism. In particular, when there is no electromagnetic effect, that is, $B \equiv 0$, the nonhomogeneous MHD system reduces to the following nonhomogeneous Navier-Stokes equations

$$\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) - \text{div}(2\mu(\rho)d) + \nabla P &= 0, \quad (NS) \\
\text{div} u &= 0.
\end{align*}$$

And we impose the following initial boundary conditions on Navier-Stokes model over a bounded domain $\Omega$.

$$u = 0, \quad \text{on } \partial \Omega \times [0, T), \quad (\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in } \Omega. \quad (4)$$
Previous work on nonhomogeneous Navier-Stokes system

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Global solution to Navier-Stokes equations

Theorem (Huang, Wang ’15)

Assume that the initial data \((\rho_0, u_0)\) satisfies the regularity condition
\[
0 \leq \rho_0 \in W^{1,q}, \quad 3 < q < \infty, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \tag{5}
\]
and the compatibility condition
\[
-\text{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) + \nabla P_0 = \rho_0^{1/2} g, \tag{6}
\]
for some \((P_0, g) \in H^1 \times L^2\), and \(0 \leq \rho_0 \leq \bar{\rho}\). Then there exists a small positive constant \(\epsilon_0\) depending on \(\Omega, q, \bar{\rho}, \bar{\mu}, \mu\) and \(\|\nabla \mu(\rho_0)\|_{L^q}\), such that if
\[
\|\nabla u_0\|_{L^2} \leq \epsilon_0,
\]
then the initial boundary value problem to (NS) admits a unique strong solution \((\rho, u, P)\), with
\[
\rho \in C([0, \infty); W^{1,q}), \quad \nabla u, P \in C([0, \infty); H^1) \cap L^2(0, \infty; W^{1,s}), 1 \leq s < q
\]
\[
\rho_t \in C([0, \infty); L^q), \quad \sqrt{\rho}u_t \in L^\infty_{loc}(0, \infty; L^2), \quad u_t \in L^2_{loc}(0, \infty; H^1_0).
\]

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Theorem (Wu ’11)  

Assume that the initial data \((\rho_0, u_0, B_0)\) satisfies the regularity condition

\[
0 \leq \rho_0 \in W^{1,q}, \quad 3 < q < \infty, \quad u_0 \in H^1_0 \cap H^2, \quad B_0 \in H^2 \tag{7}
\]

with \(\text{div} u_0 = \text{div} B_0 = 0\), and the compatibility condition

\[
-\text{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) - (B_0 \cdot \nabla)B_0 + \nabla P_0 = \rho_0^{1/2} g, \tag{8}
\]

for some \((P_0, g) \in H^1 \times L^2\). Then there exist a small time \(T\) and a unique strong solution \((\rho, u, P, B)\) to initial boundary value problem to (MHD) such that

\[
\rho \in C([0, T); W^{1,q}), \quad \nabla u, \nabla B, P \in C([0, T); H^1) \cap L^2(0, T; W^{1,s}), \quad \rho_t \in C([0, T); L^q), \quad \sqrt{\rho} u_t \in L^\infty(0, T; L^2), \quad u_t \in L^2(0, T; H^1_0), \quad B_t \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1).
\]
Global solution to MHD system

**Theorem (L.)**

Assume that the initial data \((\rho_0, u_0, B_0)\) satisfies (7)-(8), and \(0 \leq \rho_0 \leq \bar{\rho}\). Then there exists some small positive constant \(\epsilon_0\) depending on \(\Omega, q, \bar{\rho}\), \(\tilde{\mu} := \sup_{\rho_0, \rho} \mu(\rho), \underline{\mu}, \bar{\lambda} := \sup_{\rho_0, \rho} \lambda(\rho), \underline{\lambda}, \|\nabla \mu(\rho_0)\|_{L^q}\) and \(\|\nabla \lambda(\rho_0)\|_{L^q}\), such that if

\[
\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2} \leq \epsilon_0,
\]

then the initial boundary value problem to \((\text{MHD})\) admits a unique global strong solution \((\rho, u, P, B)\), with

\[
\begin{align*}
\rho & \in C([0, \infty), W^{1,q}), \quad \nabla u, \nabla B, P \in C([0, \infty), H^1) \cap L^2_{loc}(0, \infty; W^{1,s}), \\
\rho_t & \in C([0, \infty), L^q), \quad \sqrt{\rho} u_t \in L^\infty_{loc}(0, \infty; L^2), \quad u_t \in L^2_{loc}(0, \infty; H^1_0), \\
B_t & \in L^\infty_{loc}(0, \infty; L^2) \cap L^2_{loc}(0, \infty; H^1),
\end{align*}
\]

for any \(s\) with \(1 \leq s < q\).
For initial boundary problem for nonhomogeneous Navier-Stokes equations, Huang and Wang gave a sufficient condition to guarantee the global existence of strong solution. We find that the strong solution can also exist globally in time under other conditions. In particular, the smallness of $\|\nabla u_0\|_{L^2}$ can be replaced by the smallness of initial kinetic energy, where the initial kinetic energy is defined as

$$C_0 := \int_{\Omega} \frac{1}{2} \rho_0 |u_0|^2 \, dx.$$  \hspace{1cm} (9)

Now we can state our second result as follows
Theorem (L.)

For given positive numbers $\bar{\rho}$, $M$ and $N_0$, suppose that the initial data $(\rho_0, u_0)$ satisfies (5)-(6), and $0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}$, $\| \nabla u_0 \|_{L^2} \leq N_0$, and $\| \nabla \mu(\rho_0) \|_{L^q} \leq M$. Then there exists some small positive constant $\epsilon_0$, independent of $\bar{\rho}, \bar{\mu} := \sup_{[0, \bar{\rho}]} \mu(\rho), \mu, M$ and $N_0$, such that if

$$\max \left\{ \frac{\bar{\mu}}{\mu^3} \left( M_2^2 + M_4^4 M_6^6 \right) \bar{\rho}^3 N_0^2 C_0, \right.$$ 

$$M_r \bar{\rho} \frac{5r-6}{4r} \mu \frac{3(r-2)}{4r} \left( \left( 1 + \frac{\bar{\rho}}{\mu} \right) \frac{3}{4} \Theta_1 \right) \frac{1}{2} \cdot \exp\{ C \Theta_2 \}$$ 

$$+ \frac{\bar{\mu}(2r-3)/r}{\mu^3(r-1)/r} \left( M_r \bar{\rho} \right)^{5r-6} \frac{4r-6}{r} N_0 C_0 \right\} \leq \epsilon_0,$$
where $3 < r < q$, then the initial boundary value problem to (NS) admits a unique global strong solution $(\rho, u, P)$, with

$$
\rho \in C([0, \infty); W^{1,q}), \quad \nabla u, P \in C([0, \infty); H^1) \cap L^2_{loc}(0, \infty; W^{1,s}),
$$

$$
\rho_t \in C([0, \infty); L^q), \quad \sqrt{\rho} u_t \in L^\infty_{loc}(0, \infty; L^2), \quad u_t \in L^2_{loc}(0, \infty; H^1_0),
$$

for any $s$ with $1 \leq s < q$. Here,

$$
M_r = \frac{1}{\mu} + \frac{\mu}{\mu^{1/\theta_r + 1}} \cdot (4M)^{1/\theta_r}, \quad \theta_r = \frac{2r}{5r - 6} \cdot \frac{q - 3}{q},
$$

$$
\Theta_1 = \frac{M^2 \rho^6 \mu^{11/4}}{\mu^6} N_0^{11/2} C_0^{9/4} + \frac{\rho^8 \mu^{15/4}}{\mu^7} N_0^{15/2} C_0^{9/4} + \mu^{3/4} N_0^{3/2} C_0^{1/4}, \quad (11)
$$

$$
\Theta_2 = \frac{\rho^3 \mu}{\mu^5} N_0^2 C_0 + \frac{M^2 \rho^3 \mu}{\mu^3} N_0^2 C_0 + \frac{M^2 M_2^{4/3} \rho^2 \mu}{\mu} N_0^2.
$$
Remark

When $\bar{\mu} \leq C\mu$, for some $C \geq 1$, such as $\mu(\rho) = \mu + \rho^\alpha, \alpha > 0$, or $\mu(\rho) = \mu \exp \rho$, or $\mu(\rho)$ is just equal to a positive constant. We can easily see that the left hand side of (10) can be as small as desired provided $C_0$ is sufficiently small, or $\mu$ is sufficiently large, or $\bar{\rho}$ is sufficiently small.

Remark

For any given generally initial data $(\rho_0, u_0)$ containing vacuum, one also get the global strong solution of Navier-Stokes provided the lower bound of viscosity is sufficiently large. This is analogous to the well-known results due to Leray for the homogeneous incompressible Navier-Stokes equations.
Proposition

For given positive numbers $\bar{\rho}, M$ and $N_0$, assume that

$$0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}, \quad \|\nabla \mu(\rho_0)\|_{L^q} \leq M, \quad \|\nabla u_0\|_{L^2} \leq N_0. \quad (12)$$

Suppose $(\rho, u, P)$ is the unique local strong solution to (NS) on $\Omega \times [0, T]$, and if it satisfies

$$\sup_{t \in [0, T]} \|\nabla \mu(\rho(t))\|_{L^q} \leq 4M, \quad \sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2}^2 \leq 4\frac{\bar{\mu}}{\mu} N_0^2, \quad (13)$$

then one has

$$\sup_{t \in [0, T]} \|\nabla \mu(\rho(t))\|_{L^q} \leq 2M, \quad \sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2}^2 \leq 2\frac{\bar{\mu}}{\mu} N_0^2, \quad (14)$$

provided that $(10)$, together with $(11)$, holds.
Outline of proof for Navier-Stokes model

To prove the proposition, we need some a priori estimates to dominate
\[ \sup_{t \in [0, T]} \| \nabla u(t) \|_{L^2} \text{ and } \sup_{t \in [0, T]} \| \nabla \mu(\rho(t)) \|_{L^q}. \]

The estimate of \( \sup_{t \in [0, T]} \| \nabla u(t) \|_{L^2} \), to multiply the momentum
equations by \( u_t \), and integrating over \( \Omega \), after some estimates, we get

\[
\int \rho |u_t|^2 \, dx + \frac{d}{dt} \int \mu(\rho)|d|^2 \, dx
\leq C_\mu^{-1} \left( M_2^2 + M^4 M_6^6 \right) \tilde{\rho}^3 \| \nabla u \|_{L^2}^4 \cdot \int \mu(\rho)|d|^2 \, dx.
\]

Applying Gronwall’s inequality,

\[
\frac{1}{\mu} \int_0^T \int \rho |u_t|^2 \, dx + \sup_{t \in [0, T]} \| \nabla u \|_{L^2}^2
\leq \frac{\bar{\mu}}{\mu} N_0^2 \cdot \exp \left\{ C_\mu^{-1} \left( M_2^2 + M^4 M_6^6 \right) \tilde{\rho}^3 \int_0^T \| \nabla u \|_{L^2}^4 \, dt \right\}.
\]

\[
\leq 2 \frac{\bar{\mu}}{\mu} N_0^2 \quad \text{(the first term of (10) should be small)}
\]
Outline of proof for Navier-Stokes model

The estimate of \( \sup_{t \in [0, T]} \| \nabla \mu(\rho(t)) \|_{L^q} \), consider the \( x_i \) derivative of the equation for \( \mu(\rho) \),

\[
(\partial_t \mu(\rho)) + (\partial_i u \cdot \nabla) \mu(\rho) + u \cdot \nabla \partial_i \mu(\rho) = 0.
\]

It implies that for every \( t \in [0, T] \),

\[
\| \nabla \mu(\rho)(t) \|_{L^q} \leq \| \nabla \mu(\rho_0) \|_{L^q} \cdot \exp \left\{ \int_0^t \| \nabla u \|_{L^\infty} ds \right\}
\]

\[
\leq \| \nabla \mu(\rho_0) \|_{L^q} \cdot \exp \left\{ \int_0^T \| \nabla u \|_{W^{1,r}} dt \right\} 3 < r < \min \{ q, 4 \}
\]

\[
\leq \| \nabla \mu(\rho_0) \|_{L^q} \cdot \int_0^T \left( M_r \bar{\rho}^{\frac{5r-6}{4r}} \| \sqrt{\rho} u_t \|_{L^2}^{\frac{6-r}{2r}} \cdot \| \nabla u_t \|_{L^2}^{\frac{3(r-2)}{2r}}
\right.
\]

\[
+ M_r \bar{\rho}^{\frac{5r-6}{r}} \| \nabla u \|_{L^2}^{\frac{6(r-1)}{r}} \) dt
\]
Outline of proof for Navier-Stokes model

Denote \( \sigma(T) = \min\{1, T\} \), for \( T \geq 0 \), then

\[
\int_0^T \| \sqrt{\rho} u_t \|_{L^2}^{6-r} \cdot \| \nabla u_t \|_{L^2}^{2r} \, dt \\
= \int_0^{\sigma(T)} \| \sqrt{\rho} u_t \|_{L^2}^{6-r} \cdot \| \nabla u_t \|_{L^2}^{2r} \, dt + \int_{\sigma(T)}^T \| \sqrt{\rho} u_t \|_{L^2}^{6-r} \cdot \| \nabla u_t \|_{L^2}^{2r} \, dt
\]

\[
\lesssim \left( \sup_{t \in [0, \sigma(T)]} t^{\frac{5}{8}} \| \sqrt{\rho} u_t \|_{L^2} \right)^{\frac{6-r}{2r}} \left( \int_0^{\sigma(T)} t^{\frac{5}{4}} \| \nabla u_t \|_{L^2}^2 \, dt \right)^{\frac{3(r-2)}{4r}} \left( \int_{\sigma(T)}^T \left( t - \frac{5r}{2(r+6)} \right)^{\frac{r+6}{4r}} \, dt \right)^{\frac{r+6}{4r}}
\]

\[
+ \left( \sup_{t \in [\sigma(T), T]} t \| \sqrt{\rho} u_t \|_{L^2} \right)^{\frac{6-r}{2r}} \left( \int_{\sigma(T)}^T t^2 \| \nabla u_t \|_{L^2}^2 \, dt \right)^{\frac{3(r-2)}{4r}} \left( \int_{\sigma(T)}^T t^{-\frac{4r}{r+6}} \, dt \right)^{\frac{r+6}{4r}}
\]

\[
3 < r < \min\{q, 4\}
\]

\[
\lesssim \mu^{-\frac{3(r-2)}{4r}} \Theta_1^{\frac{1}{2}} \cdot \exp\{C \Theta_2\} + \mu^{-\frac{3(r-2)}{4r}} \left( \left( \frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{3}{4}} \Theta_1 \right)^{\frac{1}{2}} \cdot \exp\{C \Theta_2\}
\]
Therefore

\[ \| \nabla \mu(\rho)(t) \|_{L^q} \leq \| \nabla \mu(\rho_0) \|_{L^q} \cdot C_*(M, \bar{\rho}, \mu, \bar{\mu}, N_0, C_0) \]

Here,

\[ C_* := C \cdot M_r \bar{\rho}^{\frac{5r-6}{4r}} \mu^{-\frac{3(r-2)}{4r}} \left( \left( 1 + \frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{3}{4}} \Theta_1 \right)^{\frac{1}{2}} \cdot \exp\{ C \Theta_2 \} \]

\[ + \frac{\bar{\mu}^{(2r-3)/r}}{\mu^{3(r-1)/r}} (M_r \bar{\rho})^{\frac{5r-6}{r}} N_0^{\frac{4r-6}{r}} C_0. \]

This is why we propose the second small condition in (10).
Outline of proof for Navier-Stokes model

Since $\|\nabla \mu(\rho_0)\|_{L^q} \leq M < 4M$ and $\|\nabla u_0\|_{L^2} \leq N_0 < 2(\bar{\mu}/\mu)^{1/2}N_0$, and due to the continuity of $\nabla \mu(\rho)$ in $L^q$ and $\nabla u$ in $L^2$, there exists $T_1 \in (0, T_*)$ such that

$$\sup_{0 \leq t \leq T_1} \|\nabla \mu(\rho(t))\|_{L^q} \leq 4M, \quad \sup_{0 \leq t \leq T_1} \|\nabla u(t)\|_{L^2} \leq 2(\bar{\mu}/\mu)^{1/2}N_0.$$ 

Set

$$T^* = \sup \{ T | (\rho, u, P) \text{ is a strong solution to (NS) on } [0, T] \},$$

$$T_1^* = \sup \{ T | (\rho, u, P) \text{ is a strong solution to (NS) on } [0, T],$$

$$\sup_{0 \leq t \leq T} \|\nabla \mu(\rho(t))\|_{L^q} \leq 4M, \quad \sup_{0 \leq t \leq T} \|\nabla u(t)\|_{L^2} \leq 2(\bar{\mu}/\mu)^{1/2}N_0 \}.$$ 

Then $T_1^* \geq T_1 > 0$. Recalling above proposition, it is easy to verify

$$T^* = T_1^*,$$

We claim that $T^* = \infty$. Otherwise, assume that $T^* < \infty$. By virtue of Proposition, for every $t \in [0, T^*)$, it holds that

$$\|\nabla \rho(t)\|_{L^q} \leq 2\|\nabla \rho_0\|_{L^q}, \quad \|\nabla u(t)\|_{L^2} \leq \sqrt{2(\bar{\mu}/\mu)}N_0,$$

hence we finish the proof of Theorem.
Thank you for your attention