These are the questions I had in Spring 2004.

1 Phil Holmes: Dynamical Systems

I got a piece of paper with these two questions on it:

1. Consider the following ODE system, the claim made for its solution, and the argument given to support this claim:

\[\begin{align*}
\dot{x} &= -x + x^2y, \quad x(0) = x_0, \quad y(0) = y_0. \\
\dot{y} &= -y,
\end{align*}\]

Claim: The fixed point \((x, y) = (0, 0)\) is globally asymptotically stable (i.e. all solutions with bounded initial data approach \((0, 0)\) as \(t \to \infty\).

Argument: Clearly \(y(t) = y_0 e^{-t} \to 0\) as \(t \to \infty\), so the second (nonlinear) term of the first equation also decays to 0, leaving only the ‘stable’ linear term. Hence \(x(t) \to 0\).

Is this claim correct? If so, what is lacking in the argument and can it be turned into a proof? If the claim is not correct, can you explicitly find a solution which does not approach \((0, 0)\) as \(t \to \infty\).

2. Consider the time-dependent linear system

\[\dot{x} = A(t)x, \quad x \in \mathbb{R}^n.\]

Suppose that the time-dependent \(n \times n\) matrix \(A\) has strictly negative eigenvalues for all \(t \in \mathbb{R}\) (viewed as a parameterized family of matrices). Is the origin a stable equilibrium with no further hypotheses on \(A\)? If so, sketch a proof. If not, make suitable hypotheses so that the claim is true, or construct a counterexample.

I did the second problem. I was thinking along the lines of Jordan boxes, so I played with the matrix

\[\begin{pmatrix}
-1 & f(t) \\
0 & -1
\end{pmatrix}.\]

For \(f(t)\) constant (large, say 100 or so), it wasn’t quite good enough. The solution for \(f(t) = 100\) is

\[\begin{align*}
x(t) &= 100y_0 e^{-t} + x_0 e^{-t} \\
y(t) &= y_0 e^{-t}
\end{align*}\]

Solutions still decay to zero. In finding the solution for \(f(t) = 100\), I thought of making \(f(t)\) an exponential function, with a large enough exponent that it would last and show up in the solution. So, I did \(f(t) = e^{3t}\). Then the solution is

\[\begin{align*}
x(t) &= \frac{y_0}{3} e^{3t} + Ce^{-t} \\
y(t) &= y_0 e^{-t}
\end{align*}\]
For \( y_0 \) nonzero, \( x(t) \to \infty \).

This provides a counterexample, but the matrix norm gets very large very fast. So, we then talked about other possible conditions on the matrix.

For example, suppose the matrix is diagonal,

\[
\begin{pmatrix}
a(t) & 0 \\ 0 & b(t)
\end{pmatrix}
\]

What are conditions on \( a(t) \), \( b(t) \)? I worked through to find the actual solution, and then made the following conditions, \( a(t) \leq C_1 < 0 \), \( b(t) \leq C_2 < 0 \) (there are less strict conditions, but we stopped here).

Then, I said I could just diagonalize \( A(t) \) for each \( t \), and reduce to the diagonal case. We worked through the diagonalization procedure, and it doesn’t work as nicely as I thought it would.

Then Phil asked me to state the Poincare-Bendixson Theorem, and define the terms in the definition (\( \omega \)-limit set). Then he asked a few questions about \( \omega \)-limit sets and the Poincare-Bendixson theorem. Why is an \( \omega \)-limit set defined by a sequence of times \( t_n \) instead of just \( t \to \infty \)? etc.

## 2 Erhan Cinlar: Probability Theory

Piece of paper said: I will be asking for a general introduction to martingales: what are they, what are the basic results?

Similarly about Poisson random measures. And, if time permits, the martingales related to Brownian motion. I will not be looking for technical details, but I hope that you show a good understanding of the basic notions.

In the exam, he asked me to talk about Poisson random measures. I defined random measures and Poison random measures and talked about the Laplace functional. Then he surprised me and asked me to show the following.

Let \( N \) be a random counting measure (atomic, jumps of size 1). Suppose

\[
P\{N(A) = 0\} = e^{-\nu(A)}.
\]

Then \( N \) is a Poisson random measure.

I couldn’t prove it, so he asked me questions about martingales and Poisson processes, building up to the proof. He asked about predictable processes and martingales.
3 Bjorn Engquist: Numerical Analysis

This was the most straightforward section of my exam. The topics were straight out of his notes and the book for his course.

- Region of Absolute Stability, Implicit Euler
- Stability Conditions for PDEs (heat equation). Which are necessary? Sufficient?
- Order of Accuracy

4 End of the Exam

Professor Engquist only used 15 minutes, so the professors started asking more questions.

Professor Cinlar asked questions that were not covered in his course.
Professor Holmes and Engquist asked some questions about Green’s Functions for PDEs.