1. State the existence-uniqueness theorem for the ODE initial value problem
   \[ \dot{x} = f(x, t); \ x(0) = x_0, \]
   and sketch the main ideas of a proof.

2. Find equilibria, linearize, and discuss the stability types of the equilibria for the system:
   \[ \begin{align*}
   \dot{x}_1 &= x_2, \\
   \dot{x}_2 &= x_1 - x_1^2 + \delta x_2 - x_1 x_2.
   \end{align*} \]
   What happens as the parameter \( \delta \) varies around zero?
   Can you prove that this system has at least one, or does not have any, periodic orbits for specific values of \( \delta \)?

3. Deduce and explain the geometry of phase space for the following linear system for large \( \delta \):
   \[ \begin{align*}
   \dot{x}_1 &= -2x_1 + \delta x_2, \\
   \dot{x}_2 &= -x_2.
   \end{align*} \]
   In particular, can solutions grow in (Euclidean) norm before eventually decaying, and if so, what is their maximal ‘amplification ratio’?

4. Explain Liapunov’s direct method and how to use it to prove local and global stability of equilibria, giving specific examples.
5. Discuss the dynamics of the logistic map

\[ x_{n+1} = \lambda x_n (1 - x_n) \]

as an iterated dynamical system. What happens as the parameter \( \lambda \) varies from 0 to 4 and beyond 4?

6. State the Poincaré-Bendixson theorem and use it to show that the following ODE has at least one periodic orbit:

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_2 - x_1 x_2 - (x_1^2 + x_2^2)x_1 , \\
\dot{x}_2 &= x_1 + x_2 + x_1^2 - (x_1^2 + x_2^2)x_2 .
\end{align*}
\]

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